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Continuous rearrangement and symmetry of solutions of elliptic problems

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Abstract. This work presents new results and applications for the continuous Steiner symmetrization. There are proved some functional inequalities, e.g. for Dirichlet-type integrals and convolutions and also continuity properties in Sobolev spaces $W^{1,p}$. Further it is shown that the local minimizers of some variational problems and the nonnegative solutions of some semilinear elliptic problems in symmetric domains satisfy a weak, 'local' kind of symmetry.

Keywords. Continuous symmetrization; integral inequality; Dirichlet-type integral; semilinear elliptic problem; symmetry of the solution.

1. Introduction

Consider a variational problem of the following form

$$\text{(P)} \qquad J(v) \equiv \int_\Omega (G(x, v, |\nabla v|) - F(x, v))\mathrm{d}x \longrightarrow \text{Stat.!}, \qquad v \in K, \tag{1.1}$$

where K is a closed subset of $W_0^{1,p}(\Omega)$, $p \geq 1$, and Ω is a domain in $\mathbb{R}^n$. The *nonnegative* minimizers of problems like (P) may describe stable ('ground') states of equilibria in plasma physics, heat conduction and chemical reactors (for examples see [Di, F, K1]). We ask for symmetries of the solutions of (P), if G, F and Ω have certain 'symmetries'.

A well-known result is the following. Let $v^\star$ denote the Schwarz symmetrization of v (i.e. the radially symmetric nonincreasing rearrangement). Assume that $\Omega = \mathbb{R}^n$, $G = G(|\nabla v|)$ and G is a nonnegative and convex function with $G(0) = 0$, $F = F(v)$ and F is continuous, and K contains only nonnegative functions and has the property that, if $v \in K$, then also $v^\star \in K$. Then

$$J(v^\star) \leq J(v). \tag{1.2}$$

If, in addition, problem (P) has a *unique global* minimizer u, then we can infer from (1.2) that $u = u^\star$. (Note that this means that u is radially symmetric nonincreasing, i.e.

$$u = u(|x|) \text{ and } u \text{ is nonincreasing in } r, \quad (r = |x|).)$$

However, if the global minimizer is not unique, then the question arises whether there could be equality in (1.2) if $v \neq v^\star$. Unfortunately this case cannot be excluded, as the following simple example shows (see [BZ]).

Example 1.1. For some $p \geq 1$, let

$$J(v) := \int_{\mathbb{R}^n} |\nabla v|^p \mathrm{d}x. \tag{1.3}$$

Then there are nonnegative smooth functions v with compact support which are *not* radially symmetric and satisfy $J(v) = J(v^\star)$. Their level sets $\{v > c\},\ c > 0$, are nested, but nonconcentric balls, and the set $\{\nabla v = 0\}$ has nonempty interior, that is the graph of v has 'plateaus'.

Physically relevant are not only the global minima but also the *local* minima and *critical points* of (P). To show symmetry properties of these functions, the above argument fails, because in general the Schwarz symmetrization $v^\star$ is not close to v. Even though one expects symmetric solutions in many cases, there are again exceptions. Here is another typical example.

Example 1.2. *Semilinear problem for the p-Laplacian*: Let B be a ball in $\mathbb{R}^n$ with centre $0, f \in C(\mathbb{R}_0^+),\ p > 1$, and let $u \in C^2(\overline{B})$ satisfy

$$\begin{aligned} &-\Delta_p u \equiv -\nabla(|\nabla u|^{p-2}\nabla u) = f(u), \quad u > 0 \quad \text{in } B, \\ &u = 0 \quad \text{on } \partial B. \end{aligned} \tag{1.4}$$

Note that the associated variational problem is

$$\int_B \left(\frac{1}{p}|\nabla v|^p - F(u)\right)\mathrm{d}x \longrightarrow \text{Stat.!}, \quad v \in W_0^{1,p}(B), \tag{1.5}$$

where

$$F(v) := \int_0^v f(z)\mathrm{d}z.$$

If $p = 2$ and f is smooth then it is well-known (see [GNN]) that

$$\begin{aligned} &u = u^\star \qquad \text{and} \\ &(\partial u)/(\partial r) < 0 \quad \text{in } B \setminus \{0\}, \qquad (r = |x|). \end{aligned} \tag{1.6}$$

However, if $p > 2$ or if f is not smooth, then the conclusion (1.6) holds only under some additional assumptions. Below we give a short (but not complete) list of sufficient criteria for (1.6):

(i) $p = 2$ and $f = f_1 + f_2$, where f_1 is smooth and f_2 is increasing, [GNN];
(ii) $p = n$ and $f(v) > 0$ for $v > 0$, [KP], (see also [Lio1] for the case $p = n = 2$);
(iii) $f \in C^1(\mathbb{R}_0^+)$ and ∇u vanishes only at 0, [BaN];
(iv) $f \in C^1(\mathbb{R}_0^+)$ and $1 < p < 2$, [DamPa].

The proofs for (i), (iii) and (iv) use the so-called *moving plane method* which turned out to be a very powerful technique in proving symmetry results for positive solutions of semilinear elliptic problems in symmetric domains during the last two decades (see e.g. Se, GNN, BeN, Da, Dam, DamPa, SeZ). The moving plane technique makes essential use of the maximum principle for elliptic equations and exploits the invariance of the equation with respect to reflections. If the differential operator of the problem *degenerates* then the method is often applicable only under additional assumptions on the solution. This concerns, for instance, the p-Laplacian operator for $p > 2$ (compare the case (iii) above).

The result (ii) was proved by combining an isoperimetric inequality and a Pohozaev-type identity. However this method is not applicable if $p \neq n$.

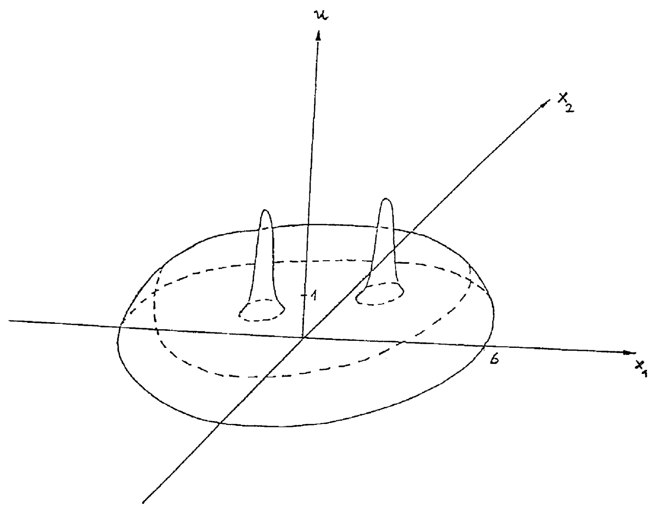

Figure 1.

One can construct radially symmetric solutions of (1.4) for which the second condition in (1.6) fails if either $p > 2$ and f is smooth, or if $p \in (1, 2]$ and f is Hölder continuous (see [GKPR]). Moreover, if $p = 2$ and f is only continuous and changes sign, then we cannot hope that the solution of (1.4) is radially symmetric. Below we give examples of solutions in the case $p \geq 2$ which have a plateau and two radially symmetric 'shifted bumps' on it. Note that similar examples can also be found in the recent paper [SeZ].

Let $p \geq 2, s > 2$,

$$w(x) = \begin{cases} (1 - |x|^2)^s & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases}, \quad \text{and}$$

$$v(x) = \begin{cases} 1 & \text{if } |x| < 5 \\ 1 - ((|x|^2 - 25)/11)^s & \text{if } 5 \leq |x| \leq 6 \end{cases}.$$

We choose $x^1, x^2 \in B_4$ with $|x^1 - x^2| \geq 2$ and set

$$u(x) := v(x) + w(x - x^1) + w(x - x^2) \qquad \forall x \in B_6.$$

The graph of u is built up by three radially symmetric 'mountains', one of them having a 'plateau' at height 1 while the other two are congruent to each other with their 'feet' lying on the plateau (see figure 1).

After a short computation we see that u is a solution of (1.4) with $\Omega = B_6$ and

$$f(u) := \begin{cases} (2s/11)^{p-1}(25 + 11(1-u)^{1/s})^{(p/2)-1}(1-u)^{p-(p/s)-1} \\ \quad \cdot \{(50/11)(p-1)(s-1) + (2ps - 2s - p + n)(1-u)^{1/s}\} \\ \hfill \text{if } 0 \leq u \leq 1, \\ (2s)^{p-1}(1 - (u-1)^{1/s})^{(p/2)-1}(u-1)^{p-(p/s)-1} \\ \quad \cdot \{-2(s-1)(p-1) + (2ps - 2s - p + n)(u-1)^{1/s}\} \\ \hfill \text{if } 1 \leq u \leq 2. \end{cases}$$

If $p = 2$ and $s > 2$ then we have $f \in C^\infty([0,2]\setminus\{1\}) \cap C^{1-(2/s)}([0,2])$. The difference quotient of f is not bounded below near $u = 1$, i.e. $f \notin C^1([0,2])$. In contrast, if $p > 2$ and $s > p/(p-2)$, then we have $f \in C^1([0,2])$.

On the other hand, the functions in the above examples are distinguished by some 'local' symmetry which can be described as follows.
(LS) Every connected component of the subset

$$\{(x, u(x)) : 0 < u(x) < \sup u,\ e \cdot \nabla u \neq 0\}$$

of the graph of u finds a congruent counterpart after reflection about some $(n-1)$-dimensional hyperplane $\{x : x \cdot e = \lambda\}$, $\lambda \in \mathbb{R}$, where e is some unit vector.

The purpose of this work is to obtain those weak symmetries for solutions of (P). The main analytic tool in the proofs will be some variant of continuous Steiner symmetrization which was developed in [B1]. Our approach is closely related to the corresponding variational problems of the differential equations. Therefore the applicability of the method seems to be restricted to equations in *divergence form*. On the other hand, we can also deal with *degenerate* elliptic operators. Furthermore, our regularity assumptions are rather mild. In most cases we only require that the solutions are differentiable in the interior of the domain and continuous up to the boundary, and the nonlinearity in the equation does not need to be smooth.

Given a Banach space X of measurable functions (e.g. $L^p(\mathbb{R}^n), p \in [1, +\infty)$), and a unit vector $e \in \mathbb{R}^n$, a *continuous Steiner symmetrization* is a continuous homotopy

$$t \longmapsto v^t, \quad 0 \leq t \leq +\infty,$$

which connects $v \in X$ with its Steiner symmetrization in direction e, v^* (see Definition 2.6 and note the difference in notation to the Schwarz symmetrization $v^\star$), such that $v^0 = v$ and $v^\infty = v^*$.

Clearly one looks for paths along which

$$J(v^t) \leq J(v), \qquad t \in [0, +\infty], \tag{1.7}$$

whenever

$$J(v^*) \leq J(v).$$

Bibliographical remarks on such homotopies were given in [B2]. Let us mention some related contributions which are connected with the *polarization* of a function or a set. This very simple kind of rearrangement was often used in the last decade to prove functional inequalities for symmetrizations (see e.g. [Du, Be, Ba] and the references cited therein).

Solynin [So] applied polarization methods to show that some capacities in the complex plane decrease under some type of continuous Steiner symmetrization. We mention that the same construction is much more simple for *convex* sets and was first used by McNabb [McN].

Finally, one can find a continuous *perturbation* of a given function (not just a homotopy!) which is formed by a certain scale of polarizations of this function (see [B3, BS]). This type of continuous rearrangement can be used to prove the symmetry of *local* minimizers for certain variational problems with potentials in a very simple manner.

Our variant of continuous symmetrization is a semigroup and satisfies the family of inequalities (1.7) for a large number of functionals, in particular for the Dirichlet-type integrals (1.3). In addition, it allows the following characterization of locally symmetric functions (see Theorem 6.2 for a more general formulation).

Let B be a ball in $\mathbb{R}^n$ with centre 0 and $p \in (1, +\infty)$. Further let $u \in W_0^{1,p}(B) \cap C^1(\overline{B})$ and $u \geq 0$. Then, if

$$\int_B (|\nabla u^t|^p - |\nabla u|^p)\mathrm{d}x = o(t) \qquad \text{as } t \searrow 0, \tag{1.8}$$

u satisfies the symmetry property (LS).

The symmetry proofs in this work depend on a number of technical steps. Let us explain the main line by considering the model equation (1.4).

Step 1. By multiplying (1.4) with $(u^t - u)$ and then by integrating we obtain

$$\int_B |\nabla u|^{p-2}\nabla u \nabla(u^t - u)\mathrm{d}x = \int_B f(u)(u^t - u)\mathrm{d}x. \tag{1.9}$$

(Note that, if $u \in W_0^{1,p}(B)$ is nonnegative, then the symmetrized functions u^t, $t \in [0, +\infty]$, also belong to $W_0^{1,p}(B)$ (see § 3), so that $(u^t - u)$ is an admissible function.)

Step 2. One shows that the right-hand side of (1.9) is of order $o(t)$ as $t \searrow 0$ (see §§ 4 and 5).

Step 3. By convexity the left-hand side of (1.9) is less than or equal to

$$\frac{1}{p}\int_B (|\nabla u^t|^p - |\nabla u|^p)\mathrm{d}x,$$

and this integral is less than or equal to 0 by (1.7) (see § 3). This yields (1.8), and so u is locally symmetric (see § 6).

Now we give an outline of our paper. In § 2 we give a new definition of the variant of continuous symmetrization which was investigated in [B2]. This new definition appears to be more transparent than the old one since it already contains the main properties of the continuous rearrangement, namely equimeasurability, monotonicity and the semi-group property. We show that open (compact) sets are transformed into open (respectively compact) sets under continuous symmetrization. At the end we recall some of the inequalities and continuity properties that we have derived in [B2] and which we will frequently use in our proofs.

The following §§ 3, 4 and 5 deal with properties of symmetrized functions in Sobolev spaces $W^{1,p}$. Most of these results are needed for the proofs of our symmetry theorems but a few of them are of independent interest in the theory of rearrangements. Those readers who are mostly interested in applications might skip these sections and return to them later. In § 3 we prove inequalities which compare some weighted norm of a nonnegative function $u \in W^{1,p}(\mathbb{R}^n)$ with the same norm of u^t. Note that similar inequalities are known for Steiner symmetrization and for the so-called starshaped rearrangements (see [K1, K4, BM] and [B4]). The proof is based on an approximation argument with a special dense subclass of piecewise smooth functions (called 'good' functions). These functions have the property that they oscillate only finitely often along any straight line lying in the direction of the symmetrization. In § 4 we show that continuous Steiner symmetrization is continuous from the right with respect to the parameter t in Sobolev spaces $W^{1,p}(\mathbb{R}^n)$, $1 \le p < +\infty$. In § 5 we study the behaviour of some nonlinear integral functionals for $t \searrow 0$ and show that it is approximately linear. In § 6 we investigate locally symmetric functions (see property (LS) above). A purely *analytic* description in terms of continuous Steiner symmetrization is given by Theorem 6.2. The preceding investigations enable us to prove that local minimizers – and also the corresponding weak solutions – of problem (P) are locally symmetric in 'symmetric' situations (Theorems 7.1–7.3 of § 7).

We point out that it is possible in many cases to derive from the *local* symmetry *additional* symmetries such as Steiner symmetry or radial symmetry (see [B4, B6]). Some further results in this direction will be published in a forthcoming paper.

2. Preliminaries

We introduce some notation. Let $\mathbb{R}^n$ be the Euclidean space, $\mathbb{R}_0^+ = [0, +\infty)$ and $\mathbb{R}^+ = (0, +\infty)$. If $n \ge 2$ and $x \in \mathbb{R}^n$, then we write

$$x = (x', y), \quad x' = (x_1, \ldots, x_{n-1}), \quad y = x_n,$$

and $|x|$ for the norm of x. $B_r(x_0)$ denotes the open ball in $\mathbb{R}^n$ with radius r centered at x_0, and we write $B_r = B_r(0)$. By ω_n we denote the volume of the n-dimensional unit ball in $\mathbb{R}^n$. For any set M in $\mathbb{R}^n$ we denote with $\overline{M}$ its closure and with $\chi(M)$ its characteristic function. If A, B are two open or compact sets then $A + B := \{z : z = x + y, x \in A, y \in B\}$ denotes their Minkowski sum. Let $\mathcal{M}(\mathbb{R}^n)$ be the set of Lebesgue measurable – measurable in short – sets in $\mathbb{R}^n$ with *finite* measure. If $M \in \mathcal{M}(\mathbb{R}^n)$ then we denote by $|M|$ its n-dimensional measure and by $S(M) = (S_1(M), \ldots, S_n(M))$ the centre of gravity where

$$S_i(M) = |M|^{-1} \int_M x_i \, \mathrm{d}x, \quad i = 1, \ldots, n.$$

We write $M \Delta N$ for the symmetric difference $(M \setminus N) \cup (N \setminus M)$ of two measurable sets M and N. Generally we treat measurable sets only in *a.e. sense*, i.e. we write

$$M = N \Longleftrightarrow |M \Delta N| = 0 \qquad \text{and}$$
$$M \subset N \Longleftrightarrow |M \setminus N| = 0.$$

If Ω is an open set in $\mathbb{R}^n$ and $p \in [1, +\infty]$ then we denote by $\|\cdot\|_p$ the usual norm in the space $L^p(\Omega)$. Sometimes we will write

$$\|u\|_{p,G} := \begin{cases} \left(\int_G |u|^p \mathrm{d}x \right)^{1/p} & \text{if } 1 \le p < +\infty \\ \operatorname{ess\,sup}_G |u| & \text{if } p = +\infty \end{cases},$$

to indicate the integration over a *subset* G of Ω. By $W^{1,p}(\Omega)$ we denote the Sobolev space of functions $u \in L^p(\Omega)$ having generalized partial derivatives $u_{x_i} \in L^p(\Omega)$, $i = 1, \ldots, n$, and we write

$$\|u\|_{W^{1,p}(\Omega)} := \|u\|_p + \sum_{i=1}^n \|u_{x_i}\|_p$$

for the norm in this space. By $W_0^{1,p}(\Omega)$ we denote the completion of $C_0^\infty(\Omega)$ under the norm $\|\cdot\|_{W^{1,p}(\Omega)}$. Recall that $W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n)$ (see [A]). By $C_0^{0,1}(\Omega)$ we denote the space of Lipschitzean functions with compact support in Ω. For any function space the subscript '$+$' denotes the corresponding subspace of nonnegative functions, e.g. $L^p_+(\Omega)$, $W^{1,p}_{0+}(\Omega)$, $C^{0,1}_{0+}(\Omega), \dots$.

A function $F : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ is called a Young function if F is continuous and convex and if $F(0) = 0$. Finally, let $\mathcal{S}(\mathbb{R}^n)$ denote the class of real measurable functions u satisfying

$$|\{x \in \mathbb{R}^n : u(x) > c\}| < +\infty \quad \forall c > \inf u.$$

Note that $L^p_+(\mathbb{R}^n)$ and $W^{1,p}_+(\mathbb{R}^n)$, $1 \le p < +\infty$, are subspaces of $\mathcal{S}_+(\mathbb{R}^n)$.

Next we give the definitions of some well-known symmetrizations.

(1) Let $M \in \mathcal{M}(\mathbb{R})$, and let M be *open or compact*. Then set

$$M^* := \begin{cases} (-(1/2)|M|, +(1/2)|M|) & \text{if } M \text{ is open} \\ [-(1/2)|M|, +(1/2)|M|] & \text{if } M \text{ is compact and } M \neq \emptyset. \end{cases} \tag{2.1}$$

If $M \in \mathcal{M}(\mathbb{R})$ is *neither open nor compact*, then M^* is given by the first formula in (2.1) in a.e. sense. M^* is called the symmetrization of M.

If $u \in \mathcal{S}(\mathbb{R})$ then the function

$$u^*(x) := \begin{cases} \sup\{c > \inf u : x \in \{u > c\}^*\} & \text{if } x \in \bigcup_{c > \inf u} \{u > c\}^* \\ \inf u & \text{if } x \notin \bigcup_{c > \inf u} \{u > c\}^* \end{cases} \tag{2.2}$$

is called the *symmetrization* or the *symmetric nonincreasing rearrangement* of u.

Note that $u(x)$ is symmetric with respect to zero, nonincreasing for $x > 0$, and we have

$$\{u > c\}^* = \{u^* > c\} \quad \forall c > \inf u. \tag{2.3}$$

(2) Let $n \ge 2$ and $M \in \mathcal{M}(\mathbb{R}^n)$. For every $x' \in \mathbb{R}^{n-1}$ we set

$$M(x') := \{y \in \mathbb{R} : (x', y) \in M\}, \quad (\text{intersection of } M \text{ with } (x', \mathbb{R})).$$

Note that every set $M \in \mathcal{M}(\mathbb{R}^n)$ has the representation

$$M = \{x = (x', y) : y \in M(x'),\ x' \in \mathbb{R}^{n-1}\},$$

where $M(x') \in \mathcal{M}(\mathbb{R})$ for almost every $x' \in \mathbb{R}^{n-1}$. The set

$$M^* := \{x = (x', y) :\ y \in (M(x'))^*, x' \in \mathbb{R}^{n-1}\} \tag{2.4}$$

is called the *Steiner symmetrization* of M with respect to y. Note that M^* is symmetric and convex with respect to the hyperplane $\{y = 0\}$. Moreover the sets $(M(x'))^*$ and thus also M^* are *pointwise* given by formula (2.4) if M is open or compact. Also it is well-known that if M is open (respectively compact) then M^* is again open (respectively compact).

If $u \in \mathcal{S}(\mathbb{R}^n)$, then the function

$$u^*(x', y) := \begin{cases} \sup\{c > \inf u : y \in \{u(x', \cdot) > c\}^*\} & \text{if } y \in \bigcup_{c > \inf u} \{u(x', \cdot) > c\}^* \\ \inf u & \text{if } y \notin \bigcup_{c > \inf u} \{u(x', \cdot) > c\}^* \end{cases} \tag{2.5}$$

is called the *Steiner symmetrization* of u. Note that $u^*(x', y)$ is symmetric with respect to $\{y = 0\}$, nonincreasing in y for $y > 0$, and we have

$$\{u(x', \cdot) > c\}^* = \{u^*(x', \cdot) > c\} \text{ for } c > \inf u \text{ and } x' \in \mathbb{R}^{n-1}. \tag{2.6}$$

(3) Let M as in (2) and let $r > 0$ satisfy $|M| = |B_r| = \omega_n r^n$. If M is open or compact, then set

$$M^\star := \begin{cases} B_r & \text{if } M \text{ is open} \\ \overline{B_r} & \text{if } M \text{ is compact and } M \neq \emptyset. \end{cases} \tag{2.7}$$

(Notice the difference between $M^\star$ and M^*!)

If M is *neither open nor compact* then $M^\star$ is given by the first formula in (2.7) in the a.e. sense. $M^\star$ is called the *Schwarz symmetrization* of M.

If $u \in \mathcal{S}(\mathbb{R}^n)$ then the function

$$u^\star(x) := \begin{cases} \sup\{c > \inf u : x \in \{u > c\}^\star\} & \text{if } x \in \bigcup_{c > \inf u} \{u > c\}^\star \\ \inf u & \text{if } x \notin \bigcup_{c > \inf u} \{u > c\}^\star \end{cases} \tag{2.8}$$

is called the *Schwarz symmetrization* or the *(radially) symmetric decreasing rearrangement* of u. Note that $u^\star$ can be written as $u^\star = u^\star(|x|)$ and is nonincreasing in $|x|$, and we have

$$\{u > c\}^\star = \{u^\star > c\} \quad \forall c > \inf u. \tag{2.9}$$

Further let us mention that for *continuous* functions u the level sets in (2.3), (2.6) and (2.9) are open such that the corresponding symmetrizations of u are *pointwise* given by these formulas. Also it is well-known that these symmetrizations are then continuous, too. Clearly for *measurable* functions the identities (2.3), (2.6) and (2.9) still hold in a.e. sense (and (2.6) for a.e. $x' \in \mathbb{R}^{n-1}$).

Remark 2.1. It is more convenient in the literature to define the symmetrizations of arbitrary measurable sets and functions *pointwise* (see e.g. [K1]). (For instance in case of the Steiner symmetrization this can be achieved by agreeing that $u^*(x', y)$ is right- (or left-) continuous in y for $y > 0$.) But it will turn out that we cannot give a *pointwise* definition of *continuous* symmetrization for *arbitrary* measurable sets and functions. Since the Steiner symmetrization will appear in that context as a special case we prefered the above settings.

This paper deals with a variant of continuous Steiner symmetrization which was introduced by the author in [B2]. Below we give a new and much shorter definition:

DEFINITION 2.1

Continuous symmetrization of sets in $\mathcal{M}(\mathbb{R})$: A family of set transformations

$$E_t : \mathcal{M}(\mathbb{R}) \longrightarrow \mathcal{M}(\mathbb{R}), \qquad 0 \le t \le +\infty,$$

satisfying the properties $(M, N \in \mathcal{M}(\mathbb{R}),\ 0 \le s, t \le +\infty)$

(i) $|E_t(M)| = |M|$, (equimeasurability),
(ii) If $M \subset N$, then $E_t(M) \subset E_t(N)$, (monotonicity),

(iii) $E_t(E_s(M)) = E_{s+t}(M)$, (semigroup property),
(iv) If $I = [y_1, y_2]$ is a bounded closed interval, then $E_t(I) = [y_1^t, y_2^t]$, where

$$\begin{aligned} y_1^t &= \tfrac{1}{2}(y_1 - y_2 + \mathrm{e}^{-t}(y_1 + y_2)), \\ y_2^t &= \tfrac{1}{2}(y_2 - y_1 + \mathrm{e}^{-t}(y_1 + y_2)), \end{aligned} \tag{2.10}$$

is called a continuous symmetrization.

Remark 2.2. One immediately verifies that the rules for the formation of symmetrized intervals (2.10) are consistent with (i)–(iii). Note also that there are possible other variants of the continuous symmetrization by modification of the formulas (2.10) (see [K2]). Some of the results in the following sections 2–4 could be proved similarly using these modified definitions. The present variant of continuous symmetrization can be used to give another *analytic* description of the symmetry property (LS) (see Theorem 6.2) which plays a central role in our approach. We underline that Theorem 6.2 is not true for the continuous symmetrization of [K2] in view of the examples given in ([B2], Remark 9). Therefore we will concentrate ourselves upon the present version.

From now on we will write for simplicity $M^t := E_t(M)$ for the symmetrized sets.

Theorem 2.1. *There exists a family of set transformations E^t, $0 \le t \le +\infty$, satisfying* (i)–(iv). *For every $M \in \mathcal{M}(\mathbb{R})$ the map $t \longmapsto M^t$, $0 \le t \le +\infty$, is a homotopy, i.e.*

$$M^0 = M, \quad M^\infty = M^*. \tag{2.11}$$

Finally, if $M \in \mathcal{M}(\mathbb{R})$ is open, then M^t has an open representative for every $t \in [0, +\infty]$.

Proof. First note that the properties (i) and (ii) imply

$$(M \cup N)^t \supset M^t \cup N^t, \tag{2.12}$$

$$(M \cap N)^t \subset M^t \cap N^t \quad \text{and} \tag{2.13}$$

$$|M \Delta N| \ge |M^t \Delta N^t|. \tag{2.14}$$

Now the proof is in several steps. Our aim is to give an explicit construction of the sets M^t, $t \in [0, +\infty]$, and to show the uniqueness of this construction.

(1) Let M be simple, that is $M = \cup_{k=1}^m I_k$, where the I_k's are disjoint bounded closed intervals. From (2.12) it follows that we must have

$$M^t \supset \bigcup_{k=1}^m I_k^t \qquad \forall t \in [0, +\infty].$$

The intervals I_k^t are disjoint for

$$t \le t_1 := \min\left\{ \log \frac{2|S(I_j) - S(I_k)|}{|I_j| + |I_k|} : 1 \le j,\ k \le m \right\},$$

and for $t = t_1$ some of them meet each other in their endpoints. In view of the equimeasurability (i) we must therefore have

$$M^t = \bigcup_{k=1}^m I_k^t \quad \text{for } 0 \le t \le t_1. \tag{2.15}$$

Furthermore, since the family M^t, $0 \le t \le +\infty$, must satisfy the semigroup property (iii), we can argue analogously for parameters $t \ge t_1$ by using the formula $M^t = (M^{t_1})^{t-t_1}$. Thus we get by induction numbers $m =: m_0 > m_1 > \cdots > m_{N-1} := 1$ and $0 =: t_0 < t_1 < \cdots < t_N := +\infty$, and bounded closed intervals $I_{k,l}$, $k = 1, \ldots, m_l$, such that for any $t \in [t_l, t_{l+1}]$ and any $l \in \{0, \ldots, N-1\}$

$$M^t = \bigcup_{k=1}^{m_l} (I_{k,l})^{t-t_l},$$

where the intervals $(I_{k,l})^{t-t_l}$ are pairwise disjoint for $t < t_{l+1}$, and where some of them coalesce for $t = t_{l+1}$. Moreover

$$|M^{t_{l+1}} \Delta M^t| \longrightarrow 0 \quad \text{as} \quad t \nearrow t_{l+1}, \quad l = 0, \ldots, N-1.$$

Finally (2.11) is satisfied. Vice versa, it is easy to see that the above construction yields a family of set transformations which satisfies (i)–(iv) in the subclass of simple sets. Furthermore, by using the rule (2.14) we check that this construction is unique. Note also, that since we may add arbitrary nullsets to the sets M^t – the above representations remain unchanged if the I_k's are *open* bounded intervals.

(2) Let M be open and $t \in [0, +\infty]$. Then we have $M = \cup_{k=1}^{+\infty} I_k$, where the I_k's are open, pairwise disjoint intervals. Setting $M_m := \cup_{k=1}^{m} I_k$, $m = 1, 2, \ldots$, we must then have $M^t \supset \cup_{m=1}^{+\infty} M_m^t$ by (2.12). (Note that the sets M_m^t are well-defined by part (1)!) Since $|M_m| = |M_m^t| \longrightarrow |M|$ as $m \to +\infty$, and since (ii) must be fulfilled, this leads to

$$M^t = \bigcup_{m=1}^{+\infty} M_m^t. \tag{2.16}$$

By using (2.14) and part (1), we check easily, that the family M^t, $0 \le t \le +\infty$, given by (2.16) does not depend on the enumeration of the intervals I_k.

Vice versa, by using again (2.14), we see that the above construction satisfies all the properties (i)–(iv) in the subclass of open sets, and that this construction is unique. In particular, formula (2.16) shows that M^t has an open representative and that (2.11) is again satisfied.

(3) Let $M \in \mathcal{M}(\mathbb{R})$. Then we have a representation

$$M = \bigcap_{n=1}^{+\infty} O_n, \tag{2.17}$$

where $O_n \supset O_{n+1}$, $n = 1, 2, \ldots$, are open sets. To satisfy (2.13), we must also have $M^t \subset \cap_{n=1}^{+\infty} O_n^t$. On the other hand, we have that $O_n^t \supset O_{n+1}^t$, $n = 1, 2, \ldots$, that is $|O_n^t| \longrightarrow |M|$ as $n \to +\infty$. The rule (i) forces the following representation,

$$M^t = \bigcap_{n=1}^{+\infty} O_n^t. \tag{2.18}$$

In view of (2.14) and part (2) we see that the set M^t given by (2.18) is independent of the representation (2.14). Furthermore, by using part (2) and once more formula (2.14), we check that the above construction satisfies the rules (i)–(iv) in the class $\mathcal{M}(\mathbb{R})$, and that this construction is the only one, satisfying these properties. Finally, (2.11) is satisfied by part (2) and (2.18). ■

Theorem 2.1 enables us to give a *pointwise* definition of the continuous symmetrization of *open* sets.

DEFINITION 2.2

Continuous symmetrization of open sets in $\mathcal{M}(\mathbb{R})$: Let $M \in \mathcal{M}(\mathbb{R})$ be open and $t \in [0, +\infty]$. Then the set

$$M^{t,O} := \bigcup \{U : U \text{ is an open representative of } N^t,\ N \text{ open},\ N \subset\subset M\} \tag{2.19}$$

is called the precise (open) representative of M^t.

One verifies easily that the above definitions of continuous symmetrization on the real axis are equivalent to those given in [B2]. Next we repeat the definition of the continuous Steiner symmetrization of [B2].

DEFINITION 2.3

Continuous (Steiner) symmetrization of sets in $\mathcal{M}(\mathbb{R}^n)$: Let $M \in \mathcal{M}(\mathbb{R}^n)$, $n \geq 2$. Then the family of sets

$$M^t := \{x = (x', y) : y \in (M(x'))^t,\ x' \in \mathbb{R}^{n-1}\}, \quad 0 \leq t \leq +\infty, \tag{2.20}$$

is called the continuous Steiner symmetrization of M. If M is open and $t \in [0, +\infty]$, then the set

$$M^{t,O} := \{x = (x', y) : y \in (M(x'))^{t,O},\ x' \in \mathbb{R}^{n-1}\} \tag{2.21}$$

is called the precise representative of M^t. Here the relation "=" in (2.21) has to be understood in the pointwise sense.

Remark 2.3. (1) Note that if $M \in \mathcal{M}(\mathbb{R}^n)$, then we have by the above definition $M^{0,O} = M$ and $M^{\infty,O} = M^*$ in the pointwise sense. (2) According to ([B2], Theorem 4) the properties listed in Theorem 2.1 remain valid for continuous Steiner symmetrization $(n \geq 2)$. Below we give three further properties:

(a) If $M, N \in \mathcal{M}(\mathbb{R}^n)$ are open sets with $M \subset N$, then

$$\operatorname{dist}\{M; \partial N\} \leq \operatorname{dist}\{M^{t,O}; \partial N^{t,O}\} \qquad \forall 0 \leq t \leq +\infty. \tag{2.22}$$

It is easy to verify (compare also [BS]) that (2.22) yields the following:

(b) *Smoothing property:* If $M \in \mathcal{M}(\mathbb{R}^n)$, $t \in [0, +\infty]$ and $r > 0$, then

$$M^{t,O} + B_r \subset (M + B_r)^{t,O} \qquad \text{and} \tag{2.23}$$

$$M^{t,O} \setminus (\partial M^{t,O} + \overline{B_r}) \supset (M \setminus (\partial M + \overline{B_r}))^{t,O}. \tag{2.24}$$

From the Definitions 2.1–2.3 we immediately derive the following property:

(c) *Continuity from the inside:* If $\{M_k\}$ is an increasing sequence of open sets with $|\cup_{k=1}^{+\infty} M_k| < +\infty$, then

$$\bigcup_{k=1}^{+\infty} (M_k)^{t,O} = \left(\bigcup_{k=1}^{+\infty} M_k\right)^{t,O} \qquad \forall t \in [0, +\infty]. \tag{2.25}$$

Theorem 2.1 suggests that if $M \in \mathcal{M}(\mathbb{R}^n)$ is open then the precise representatives $M^{t,O}$, $t \in [0,+\infty]$, should be open too. It was kindly pointed out to me by Buttazzo that this fact is missing in [B2]. Nevertheless its proof is simple and requires no more than the monotonicity (Definition 2.1(ii)) and the properties (b) and (c) from Remark 2.3. This observation was first made by Sarvas [Sa] in a context of general rearrangements.

Lemma 2.1. *Let $M \in \mathcal{M}(\mathbb{R}^n)$ and open. Then the sets $M^{t,O}$, $0 \leq t \leq +\infty$, are open too.*

Proof. We fix $t \in [0,+\infty]$. In view of (2.25) we have

$$M^{t,O} = \bigcup_{k=1}^{+\infty} (M \setminus (\partial M + (1/k)\overline{B_1}))^{t,O}.$$

By the monotonicity (Definition 2.1(ii)) and by (2.24) this yields

$$M^{t,O} = \bigcup_{k=1}^{+\infty} (M^{t,O} \setminus (\partial M^{t,O} + (1/k)\overline{B_1})),$$

which means that $M^{t,O}$ is open. The lemma is proved. ■

Remark 2.4. Similarily as in the case of the Steiner and Schwarz symmetrization it is also possible to give the continuous symmetrization of *compact sets* a pointwise meaning (see [B4]). But since we do not need such a construction in this paper we omit the details.

From now on let us agree that if we speak about the continuous symmetrization of *open* sets, then we always mean their precise representatives, and we omit the superscript O.

DEFINITION 2.4

Continuous (Steiner) symmetrization of functions: Let $u \in \mathcal{S}(\mathbb{R}^n)$. Then the family of functions u^t, $0 \leq t \leq +\infty$, defined by

$$u^t(x) := \begin{cases} \sup\{c > \inf u : x \in \{u > c\}^t\} & \text{if } x \in \bigcup_{c > \inf u} \{u > c\}^t \\ \inf u & \text{if } x \notin \bigcup_{c > \inf u} \{u > c\}^t \end{cases}, \quad x \in \mathbb{R}^n, \tag{2.26}$$

is called continuous (Steiner) symmetrization of u with respect to y in the case $n \geq 2$ and continuous symmetrization in the case $n = 1$.

Remark 2.5. It is easy to see that formula (2.26) is equivalent to the following relations

$$\begin{aligned} \{u^t > c\} &= \{u > c\}^t \qquad \forall c > \inf u, \\ \{u^t = \inf u\} &= \mathbb{R}^n \setminus \bigcup_{c > \inf u} \{u > c\}^t, \\ \{u^t = +\infty\} &= \bigcap_{c > \inf u} \{u > c\}^t. \end{aligned} \tag{2.27}$$

It was shown in [B2], that $u^0 = u$ and $u^\infty = u^*$. Furthermore, if u is continuous, then for every $t \in [0,+\infty]$ the function u^t has a continuous representative which is given by the formulas (2.26) and (2.27) in pointwise sense and on the right-hand sides of these

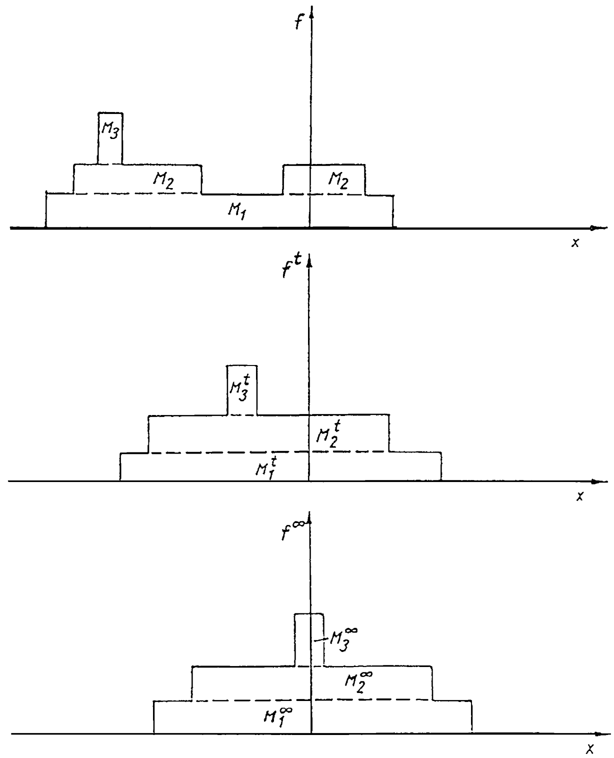

Figure 2.

formulas are taken the precise (open) representatives of the corresponding level sets. One can illustrate the formulas (2.26), (2.27) by continuously rearranging a step function as in figure 2.

If

$$u = c_0 + \sum_{i=}^{m} c_i \chi(M_i), \tag{2.28}$$

where $M_1 \supset \cdots \supset M_m,\ M_i \in \mathcal{M}(\mathbb{R}^n)$, and $c_0 \in \mathbb{R},\ c_i > 0,\ i = 1, \ldots, m$, then

$$u^t = c_0 + \sum_{i=1}^{m} c_i \chi(M_i^t), \qquad t \in [0, +\infty]. \tag{2.29}$$

From now on let us agree that if we speak about the continuous symmetrizations of *continuous* functions, then we always mean their precise (continuous) representatives.

Remark 2.6. Let us recall some properties of continuous symmetrization that we proved in [B2] and which we will use from time to time $(M, N \in \mathcal{M}(\mathbb{R}^n), u, v, w \in \mathcal{S}_+(\mathbb{R}^n)$, $t \in [0, +\infty])$.

(1) *Monotonicity* (see [B2], Theorem 5):

If $u \le v$, then

$$u^t \le v^t. \tag{2.30}$$

(2) *Cavalieri's principle* (see [B2], Theorem 8):

$$\int_{\mathbb{R}^n} F(u)\mathrm{d}x = \int_{\mathbb{R}^n} F(u^t)\mathrm{d}x, \tag{2.31}$$

if F is Borel measurable and the left-hand side of (2.31) converges.

(3) *Continuity with respect to the parameter t:* If $t_m \to t$ as $m \to +\infty$, then (see [B2], Theorem 3)

$$M^{t_m} \longrightarrow M^t \qquad \text{in measure,} \tag{2.32}$$

and if u is a.e. finite, then (see [B2], Theorem 7)

$$u^{t_m} \longrightarrow u^t \qquad \text{in measure.} \tag{2.33}$$

(4) *Centre-formula* (see [B2], Remark 4):

$$S(M^t) = (S_1(M), \ldots, S_{n-1}(M), e^{-t}S_n(M)). \tag{2.34}$$

(5) *Nonexpansivity in* $L^p(\mathbb{R}^n)$, $1 \le p \le +\infty$, (see [B2], Lemma 3): If $u, v \in L^p(\mathbb{R}^n)$, then

$$\|u^t - v^t\|_p \le \|u - v\|_p. \tag{2.35}$$

(6) *Hardy–Littlewood inequality* (see [B2], Lemma 4): If $u, v \in L^2(\mathbb{R}^n)$, then

$$\int_{\mathbb{R}^n} u^t v^t \,\mathrm{d}x \ge \int_{\mathbb{R}^n} uv \,\mathrm{d}x. \tag{2.36}$$

(7) If u is Lipschitz continuous with Lipschitz constant L then u^t is Lipschitz continuous, too, with Lipschitz constant less or equal to L, (see [B2], Theorem 7).

Note that (1), (2), (5) and (6) are common properties of monotone equimeasurable rearrangements (see [K1, BS]). We mention that the Lipschitz continuity is in fact the 'best' regularity which is preserved under continuous symmetrization. This can be seen by symmetrizing a function $f \in C^1(\mathbb{R})$ which has more than two monotonicity intervals (see figure 3). The functions f^t and f^∞ are *not differentiable* in the marked points.

From now on we will assume that $n \ge 2$. Since the continuous Steiner symmetrization is in fact a 'one-dimensional' construction, the results of this work can be transferred to the simpler case $n = 1$ with obvious changes.

3. Dirichlet-type inequalities

In this section we prove various inequalities which compare some (weighted) Sobolev norm of a function u with the same norm of u^t. The strategy in the proofs consists in changing locally the variable of integration from y to u in the functionals. Functions for which this is possible are characterized by the following:

DEFINITION 3.1 ('Good' functions)

A function u is called good if u is defined on $\mathbb{R}^n$ and nonnegative, piecewise smooth with compact support, if for every $x' \in \mathbb{R}^{n-1}$ and $c > 0$ the equation $u(x', y) = c$ has only a finite number of solutions $y = y_k$, $k = 1, \ldots, l$, and if

$$\inf\{|u_y(x)| : x \in \mathbb{R}^n, u_y(x) \text{ exists}\} > 0. \tag{3.1}$$

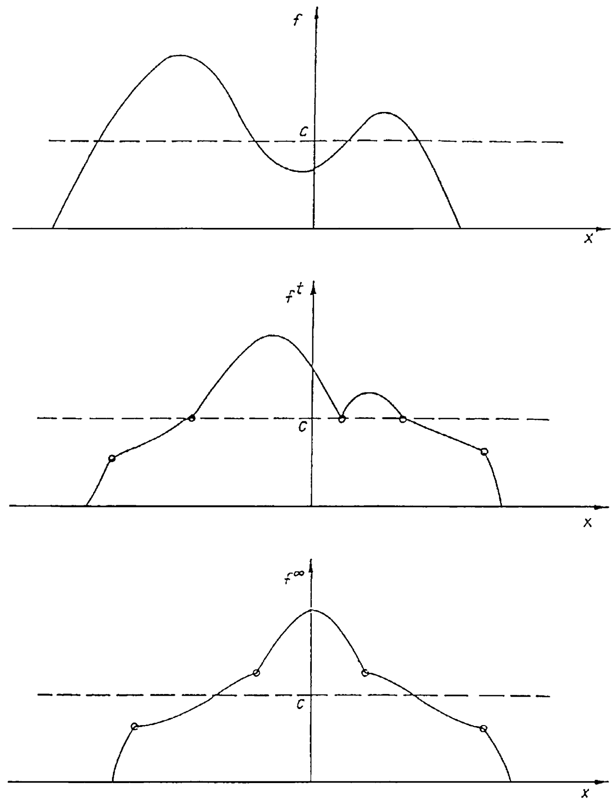

Figure 3.

Remark 3.1. (1) Good functions are dense in $W_+^{1,p}(\mathbb{R}^n)$ in the norm of $W^{1,p}(\mathbb{R}^n)$ for every $p \in [1, +\infty)$. This can be seen as follows. $C_0^\infty(\mathbb{R}^n)$ is dense in $W^{1,p}(\mathbb{R}^n)$. Any C_0^∞-function can be approximated by piecewise linear functions with compact support. If u is piecewise linear with support in $B_R(0)$, $(R > 0)$, then set

$$\varepsilon_0 := \min\{|u_y(x)| : x \in \mathbb{R}^n, u_y(x) \text{ exists and is } \neq 0\} > 0$$

and

$$v(x) := \begin{cases} 1 - R^{-1}|y| & \text{if } |y| < R \\ 0 & \text{if } |y| \geq R \end{cases}.$$

The functions $u_\varepsilon := u + \varepsilon v$ are good if $0 < |\varepsilon| < \varepsilon_0$, and u_ε converges to u in $W^{1,p}(\mathbb{R}^n)$ as ε tends to zero. Note that the same argumentation can be found in ([K1], pp. 49) for good piecewise linear functions in $W_0^{1,p}(\Omega)$, where Ω is a bounded domain. (2) Let $u \in W_+^{1,1}(\mathbb{R}^n) \cap C_{\text{loc}}^1(\mathbb{R}^n)$. Then u is absolutely continuous on almost every line $\{x' = \text{const}\}$ (see [EG], p. 164). From this we can infer (compare [C], Appendix 1 and 4) that u is 'generically' good, i.e. for *almost every* pair $(x', c) \in \mathbb{R}^{n-1} \times \mathbb{R}_0^+$ the equation $u(x', y) = c$ has only a finite number of solutions, and the equation $u_y(x', y) = 0$ does not have any solution. (3) If u is good and piecewise linear, then the functions u^t, $t \in [0, +\infty]$, are in general not piecewise linear, as one can see from simple examples of functions which are not quasiconcave in the direction y. But there holds the following:

Lemma 3.1. *Let u be good. Then the functions u^t, $t \in [0, +\infty]$, are good, too.*

Proof. The set

$$K := \{(x', u) \in \mathbb{R}^{n-1} \times \mathbb{R}^+ : \exists (x', y) \in \operatorname{supp} u, \text{ such that } u = u(x', y)\} \tag{3.2}$$

is compact. Furthermore, it is easy to see that for a.e. point $(x'_0, u_0) \in K$ there exists an open neighbourhood $V \subset K$ such that the equation $u = u(x', y)$ has *exactly* $2m$, $(m = m(V))$, solutions $y = y_k(x', u)$ in V, $y_k \in C^1(V)$, $k = 1, \ldots, 2m$, and such that $y_1 < \cdots < y_{2m}$. Thus u can be represented in V by *local* inverse functions $y = y_k(x', u)$, and we have that

$$u_y(x', y_k) = \left(\frac{\partial y_k}{\partial u}\right)^{-1} \begin{cases} > 0 & \text{if } k \text{ is odd} \\ < 0 & \text{if } k \text{ is even} \end{cases},$$
$$u_{x_i}(x', y_k) = -\frac{\partial y_k}{\partial x_i}\left(\frac{\partial y_k}{\partial u}\right)^{-1}, \quad i = 1, \ldots, n-1, \quad (k = 1, \ldots, 2m). \tag{3.3}$$

Our aim is to derive analogous representations for u^t, $t \in [0, +\infty]$. To this end we restrict our considerations to one of the above open sets V. First observe that for each $(x', u) \in V$ the equation $u = u^t(x', y)$ has at most $2m$ solutions y, by the proof of Theorem 2.1. Let V' be an open set with $V' \subset\subset V$. Then we see from Definitions 2.1–2.3 that for small t, u^t can be represented in V' by smooth inverse functions $y = y_k^t(x', u)$ through the formulas

$$y_{2k-1}^t = \frac{1}{2}(y_{2k-1} - y_{2k} + \mathrm{e}^{-t}(y_{2k-1} + y_{2k})),$$
$$y_{2k}^t = \frac{1}{2}(y_{2k} - y_{2k-1} + \mathrm{e}^{-t}(y_{2k-1} + y_{2k})), \tag{3.4}$$

and there hold the following identities,

$$u_y^t(x', y_k^t) = \left(\frac{\partial y_k^t}{\partial u}\right)^{-1} \begin{cases} > 0 & \text{if } k \text{ is odd} \\ < 0 & \text{if } k \text{ is even} \end{cases},$$
$$u_{x_i}^t(x', y_k^t) = -\frac{\partial y_k^t}{\partial x_i}\left(\frac{\partial y_k^t}{\partial u}\right)^{-1}, \quad i = 1, \ldots, n-1, \quad (k = 1, \ldots, 2m). \tag{3.5}$$

Suppose that $t_1 (= t_1(x', u))$ is the first value of t, such that some of the intervals $[y_{2k-1}^t(x', u), y_{2k}^t(x', u)]$, $k = 1, \ldots, m$, coalesce. Note that $t_1(x', u)$ varies continuously in V'. For simplicity in notation let us assume that we have $y_{2k}^{t_1} = y_{2k+1}^{t_1}$, $k = 1, \ldots, l$, $(l \le m-1)$, at some point $(x'_0, u_0) \in V'$. Following the proof of Theorem 2.1, we see that there is some (small) neighbourhood V'' of (x'_0, u_0) and some number $t_2 > \sup\{t_1(x', u): (x', u) \in V''\}$, such that the functions $y_1^t(x', u)$ and $y_{2l}^t(x', u)$, $(t \le t_1(x', u))$, find 'continuations' $\eta_i^t(x', u)$, for $t_1(x', u) < t < t_2$, and such that $u = u^t(x', \eta_i^t)$, $i = 1, 2$, in V''. From the equimeasurability we have that

$$\eta_2^t - \eta_1^t = \sum_{k=1}^{l}(y_{2k-1} - y_{2k}) = y_{2l}^{t_1} - y_1^{t_1}. \tag{3.6}$$

Since the set $\cup_{k=1}^{l}[y_{2k-1}, y_{2k}]$ is continuously symmetrized independently from the other intervals for $t < t_2$, we may apply the centre formula (2.34) onto this set. Together with the semigroup property this yields

$$\frac{\eta_1^t + \eta_2^t}{2} = \mathrm{e}^{t-t_1} \frac{\sum_{k=1}^{l} \frac{1}{2}(y_{2k}^{t_1} + y_{2k-1}^{t_1})}{\sum_{k=1}^{l}(y_{2k}^{t_1} - y_{2k-1}^{t_1})}. \tag{3.7}$$

After a differentiation with respect to x_i, $i = 1, \dots, n-1$, we infer from (3.6), (3.7) that

$$\begin{aligned} \eta_{2,x_i}^t - \eta_{1,x_i}^t &= \sum_{k=1}^{l}(y_{2k,x_i}^{t_1} - y_{2k-1,x_i}^{t_1}), \\ \eta_{2,x_i}^t + \eta_{1,x_i}^t &= \frac{2\sum_{k=1}^{l}(y_{2k}^{t_1} y_{2k,x_i}^{t_1} - y_{2k-1}^{t_1} y_{2k-1,x_i}^{t_1})}{\sum_{k=1}^{l}(y_{2k}^{t_1} - y_{2k-1}^{t_1})} \\ &\quad - \frac{\sum_{k=1}^{l}((y_{2k}^{t_1})^2 - (y_{2k-1}^{t_1})^2)}{(\sum_{k=1}^{l}(y_{2k}^{t_1} - y_{2k-1}^{t_1}))^2} \sum_{k=1}^{l}(y_{2k,x_i}^{t_1} - y_{2k-1,x_i}^{t_1}). \end{aligned} \tag{3.8}$$

In view of the equalities

$$\begin{aligned} &y_{2k-1}^{t_1} = y_{2k}^{t_1}, \quad k = 1, \dots, l, \\ &y_{2l}^{t_1} = \eta_2^{t_1}, \quad y_1^{t_1} = \eta_1^{t_1}, \end{aligned} \tag{3.9}$$

we obtain from (3.8)

$$\begin{aligned} \eta_{1,x_i}^{t_1} &= -\frac{1}{y_{2l}^{t_1} - y_1^{t_1}} \sum_{j=1}^{2l}(-1)^j y_{j,x_i}^{t_1}(y_{2l}^{t_1} - y_j^{t_1}), \\ \eta_{2,x_i}^{t_1} &= \frac{1}{y_{2l}^{t_1} - y_1^{t_1}} \sum_{j=1}^{2l}(-1)^j y_{j,x_i}^{t_1}(y_j^{t_1} - y_1^{t_1}). \end{aligned} \tag{3.10}$$

Analogously we compute the derivatives with respect to u as

$$\begin{aligned} |\eta_{1,u}^{t_1}| &= \frac{1}{y_{2l}^{t_1} - y_1^{t_1}} \sum_{j=1}^{2l} |y_{j,u}^{t_1}|(y_{2l}^{t_1} - y_1^{t_1}), \\ |\eta_{2,u}^{t_1}| &= \frac{1}{y_{2l}^{t_1} - y_1^{t_1}} \sum_{j=1}^{2l} |y_{j,u}^{t_1}|(y_j^{t_1} - y_1^{t_1}), \end{aligned} \tag{3.11}$$

and we have $\eta_{2,u}^{t_1} < 0 < \eta_{1,u}^{t_1}$. We will not specify formulas for the 'future' of the remaining intervals $[y_{2k-1}^{t_1}(x', u), y_{2k}^{t_1}(x', u)]$, $k = l+1, \dots, m$, for $t \in (t_1(x', u), t_2)$. Some of these intervals might be computed henceforth according to (3.4) while others coalesce during that time. This leads to analogous computations.

By means of the semigroup property we can repeat these considerations step by step for every $t \in [0, +\infty]$ and for almost every points of K. From the formulas (3.4), (3.5), (3.10) and (3.11) we see that u^t is piecewise smooth and

$$\inf\{|u_y^t(x)| : x \in \mathbb{R}^n,\ u_y^t(x) \text{ exists}\} > 0, \quad t \in (0, +\infty].$$

The lemma is proved. ■

Theorem 3.1. *Let be u a good function, G a Young function and $a \in C(\mathbb{R})$ nonnegative, even and convex. Then for every $t \in [0, +\infty]$*

$$\int_{\mathbb{R}^n} G\left(\left\{a^2(y)\left(\frac{\partial u}{\partial y}\right)^2 + \sum_{i=1}^{n-1}\left(\frac{\partial u}{\partial x_i}\right)^2\right\}^{1/2}\right)\mathrm{d}x$$
$$\geq \int_{\mathbb{R}^n} G\left(\left\{a^2(y)\left(\frac{\partial u^t}{\partial y}\right)^2 + \sum_{i=1}^{n-1}\left(\frac{\partial u^t}{\partial x_i}\right)^2\right\}^{1/2}\right)\mathrm{d}x. \tag{3.12}$$

Proof. We use the notations of the previous proof. We may change locally the variable of integration in (3.12) from (x', y) to (x', u). Then the integrals on the left and right-hand side of (3.12) become $\int_K I(x', u)\mathrm{d}x'\mathrm{d}u$ and $\int_K I_t(x', u)\mathrm{d}x'\mathrm{d}u$, respectively, where K is given by (3.2) and I and I_t are nonnegative functions which will be specified below. Then, to prove (3.12), it is sufficient to show that

$$I(x', u) \geq I_t(x', u) \quad \text{for a.e. } (x', u) \in K. \tag{3.13}$$

Using the notations of the previous proof, we compute

$$I(x_0', u_0) = \sum_{k=1}^{2m} G\left(\left(\left|\frac{\partial y_k}{\partial u}\right|\right)^{-1}\left\{a^2(y_k) + \sum_{i=1}^{n-1}\left(\frac{\partial y_k}{\partial x_i}\right)\right\}^{1/2}\right)\left|\frac{\partial y_k}{\partial u}\right|. \tag{3.14}$$

Similarly, we have that

$$I_t(x_0', u_0) = \sum_{k=1}^{2m} G\left(\left(\left|\frac{\partial y_k^t}{\partial u}\right|\right)^{-1}\left\{a^2(y_k^t) + \sum_{i=1}^{n-1}\left(\frac{\partial y_k^t}{\partial x_i}\right)\right\}^{1/2}\right)\left|\frac{\partial y_k^t}{\partial u}\right|, \quad \text{for } t \in (0, t_1), \tag{3.15}$$

where $t_1 = t_1(x_0', u_0)$. Thus, to prove (3.13) at (x_0', u_0) for $t \in (0, t_1)$, it suffices to show that

$$\varphi_k(t) := \sum_{l=2k-1}^{2k} G\left(\left(\left|\frac{\partial y_l^t}{\partial u}\right|\right)^{-1}\left\{a^2(y_l^t) + \sum_{i=1}^{n-1}\left(\frac{\partial y_l^t}{\partial x_i}\right)^2\right\}^{1/2}\right)\left|\frac{\partial y_l^t}{\partial u}\right|$$
$$\text{is nondecreasing for } t \in [0, t_1),\ (k = 1, \ldots m). \tag{3.16}$$

To see this, we formally extend the definition (3.4) of the functions y_k^t, $(k = 1, \ldots, 2m)$, for all $t \in [0, +\infty]$. We introduce the new parameter $\lambda := (1/2)(1 - \mathrm{e}^{-t})$, and set $\psi_k(\lambda) := \varphi_k(t)$. By setting in addition

$$\psi_k(1 - \lambda) := \psi_k(\lambda) \qquad \forall \lambda \in [0, (1/2)],$$

a simple calculation shows that $\psi_k(\lambda)$, $\lambda \in [0, 1]$, is convex. This proves (3.16).

Next assume that at the moment $t = t_1$ the intervals $[y_{2k-1}^t, y_{2k}^t]$, $k = 1, \ldots, l$, $(l \leq m)$, coalesce and are 'continued' in a single interval $[\eta_1^t, \eta_2^t]$ according to the formulas (3.6), (3.7). Note that $I_t(x_0', u_0)$ is not defined at $t = t_1$. Setting $y_k := y_k^{t_1}$, $k = 1, \ldots, 2l$, and $\eta_k := \eta_k^{t_1}$, $k = 1, 2$, we want to show that

$$\sum_{k=1}^{2} G\left((|\eta_{k,u}|)^{-1}\left\{a^2(\eta_k) + \sum_{i=1}^{n-1}(\eta_{k,x_i})^2\right\}^{1/2}\right)|\eta_{k,u}|$$

$$\leq \sum_{k=1}^{2l} G\left((|y_{k,u}|)^{-1}\left\{a^2(y_k)+\sum_{i=1}^{n-1}(y_{k,x_i})^2\right\}^{1/2}\right)|y_{k,u}|. \tag{3.17}$$

Choosing

$$\lambda_j := \frac{|y_{j,u}|(y_j-y_1)}{\sum_{j=1}^{2l}|y_{j,u}|(y_j-y_1)}, \quad \mu_j := \frac{|y_{j,u}|(y_{2l}-y_j)}{\sum_{j=1}^{2l}|y_{j,u}|(y_{2l}-y_j)} \quad \text{and}$$

$$z_j := \left\{a^2(y_j)+\sum_{i=1}^{n-1}(y_{j,x_i})^2\right\}^{1/2}(|y_{j,u}|)^{-1}, \quad j=1,\dots,2l,$$

the right-hand side of (3.17) becomes

$$\sum_{j=1}^{2l}\left(\frac{\sum_{k=1}^{2l}|y_{k,u}|(y_k-y_1)}{y_{2l}-y_1}\lambda_j F(z_j)+\frac{\sum_{k=1}^{2l}|y_{k,u}|(y_{2l}-y_k)}{y_{2l}-y_1}\mu_j F(z_j)\right)=: I.$$

Since G is convex we conclude from this

$$I \geq G\left(\sum_{j=1}^{2l}\lambda_j z_j\right)\frac{\sum_{k=1}^{2l}|y_{k,u}|(y_k-y_1)}{y_{2l}-y_1}+G\left(\sum_{j=1}^{2l}\mu_j z_j\right)\frac{\sum_{k=1}^{2l}|y_{k,u}|(y_{2l}-y_k)}{y_{2l}-y_1}=: I'.$$

Furthermore, from the monotonicity and convexity of the function $\varphi(\xi_1,\dots,\xi_n) := \{\xi_1^2+\cdots\xi_n^2\}^{1/2}$ we derive

$$\sum_{j=1}^{2l}\lambda_j z_j \geq \frac{\{(y_{2l}-y_1)^2a^2(y_{2l})+\sum_{i=1}^{n-1}[\sum_{j=1}^{2l}(-1)^j y_{j,x_i}(y_j-y_1)]^2\}^{1/2}}{\sum_{k=1}^{2l}|y_{k,u}|(y_k-y_1)}$$

and

$$\sum_{j=1}^{2l}\mu_j z_j \geq \frac{\{(y_{2l}-y_1)^2a^2(y_1)+\sum_{i=1}^{n-1}[\sum_{j=1}^{2l}(-1)^j y_{j,x_i}(y_{2l}-y_j)^2]^2\}^{1/2}}{\sum_{k=1}^{2l}|y_{k,u}|(y_{2l}-y_k)}.$$

Together with the monotonocity of G this yields

$$\begin{aligned} I' \geq {} & \frac{\sum_{j=1}^{2l}|y_{j,u}|(y_j-y_1)}{y_{2l}-y_1} \\ & \times G\left(\frac{\{(y_{2l}-y_1)^2a^2(y_{2l})+\sum_{i=1}^{n-1}[\sum_{j=1}^{2l}(-1)^j y_{j,x_i}(y_j-y_1)]^2\}^{1/2}}{\sum_{j=1}^{2l}|y_{j,u}|(y_j-y_1)}\right) \\ & +\frac{\sum_{j=1}^{2l}|y_{j,u}|(y_{2l}-y_j)}{y_{2l}-y_1} \\ & \times G\left(\frac{\{(y_{2l}-y_1)^2a^2(y_1)+\sum_{i=1}^{n-1}[\sum_{j=1}^{2l}(-1)^j y_{j,x_i}(y_{2l}-y_j)]^2\}^{1/2}}{\sum_{j=1}^{2l}|y_{j,u}|(y_{2l}-y_j)}\right). \end{aligned}$$

But in view of the identities (3.9)–(3.11) this last term is equal to the left-hand side of (3.17). Now from the inequalities (3.16) and (3.17) we obtain easily that the function $h(t) := I_t(x_0', u_0)$, $t \in [0, t_2)$, does not increase across the value $t = t_1$. Moreover, using the semigroup property we see that $h(t)$, $t \in [0, +\infty]$, is well-defined – with the except of a finite number of values t – and nonincreasing.

By Lemma 3.1 we can argue similarly for a.e. $(x', u) \in K$. This shows (3.13), and the theorem is proved. ■

A slight generalization of the previous Theorem 3.1 is the following:

COROLLARY 3.1

Let the functions $G(x', v, z)$, $a(x', y, v)$, $a_{ij}(x', v)$, $i, j = 1, \ldots, n-1$, *be continuous* $\forall (x, v, z) \in \mathbb{R}^n \times (\mathbb{R}_0^+)^2$. *Let G be nonnegative and convex in z with* $G(x', v, 0) = 0$ $\forall (x', v) \in \mathbb{R}^{n-1} \times \mathbb{R}_0^+$. *Further on let a be positive, even and convex in y, and let the matrix* (a_{ij}) *be positive definite. Finally let u be a good function. Then for every* $t \in [0, +\infty]$ *we have*

$$\int_{\mathbb{R}^n} G\left(x', u, \left\{a^2 u_y^2 + \sum_{i,j=1}^{n-1} a_{ij} u_{x_i} u_{x_j}\right\}^{1/2}\right) \mathrm{d}x$$
$$\geq \int_{\mathbb{R}^n} G\left(x', u, \left\{\tilde{a}^2 (u_y^t)^2 + \sum_{i=1}^{n-1} \tilde{a}_{ij} u_{x_i}^t u_{x_j}^t\right\}^{1/2}\right) \mathrm{d}x. \tag{3.18}$$

(*For simplicity we wrote* $u = u(x)$, $u^t = u^t(x)$, $a = a(x, u(x))$, $\tilde{a} = a(x, u^t(x))$ *and* $a_{ij} = a_{ij}(x', u(x))$, $\tilde{a_{ij}} = a_{ij}(x', u^t(x))$, $i, j = 1, \ldots, n-1$, *in* (3.18).)

Proof. We fix an arbitrary point $(x_0', u_0) \in \mathbb{R}^{n-1} \times \mathbb{R}_0^+$. From the previous proof we see that it is sufficient to show the statements (3.16) and (3.17) at (x_0', u_0) – with the terms containing partial derivatives in x_i, $(i = 1, \ldots, n-1)$, replaced by some corresponding quadratic forms. For an appropriate linear mapping $x' \mapsto \xi' \in \mathbb{R}^{n-1}$ we can achieve that the function $v(\xi, y) := u(x', y)$ satisfies

$$\sum_{i=1}^{n-1} \tilde{a}_{ij} u_{x_i} u_{x_j} = \sum_{i=1}^{n-1} \left(\frac{\partial v}{\partial \xi_i}\right)^2$$

at the point (x_0', u_0). Since, by the definition of the continuous symmetrization, $v^t(\xi', y) = u^t(x', y)$, $(t \in [0, +\infty])$, (3.16) and (3.17) then follow as before. ■

Remark 3.2. (1) Integrals as in (3.12) and (3.18) with $G = G(z) = z^2$ and a some power of y appear in variational problems for two-dimensional or axisymmetric flows (see [F, B1]). (2) In the case $t = +\infty$ (i.e. for Steiner symmetrization) the inequality (3.18) can be proved in a simpler manner (see [B5]). Equation (3.18) seems to be the most general Dirichlet-type inequality for Steiner symmetrization which appeared in the literature. Note that some similar inequalities with a radial weight function in the integrand are well known for the so-called starshaped rearrangements (see [BM, K1, 3, 4 and M]).

If $a = a_i \equiv 1$ and $G(z) = z^p$ for some $p \in (1, +\infty)$ in (3.18) then we are led to norm inequalities in $W^{1,p}$. For the proof we further need the following nice equivalence principle for convex inequalities which was shown in ([ALT], Corollary 3.1).

Lemma 3.2. *Let* $u, v \in \mathcal{S}_+(\mathbb{R}^n)$. *Then the following two properties* (i) *and* (ii) *are equivalent to each other,*

(i) $$\int_{\mathbb{R}^n} G(u)\mathrm{d}x \le \int_{\mathbb{R}^n} G(v)\mathrm{d}x, \quad \text{for every Young function } G.$$

(ii) $$\int_M u\,\mathrm{d}x \le \sup\left\{\int_N v\,\mathrm{d}x : |N| \le |M|,\ N \in \mathcal{M}(\mathbb{R}^n)\right\},$$
for every set $M \in \mathcal{M}(\mathbb{R}^n)$.

Theorem 3.2. *Let $u \in W^{1,p}(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$ for some $p \in [1, +\infty]$. Then for every $t \in [0, +\infty]$ we have $u^t \in W^{1,p}(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$ and*

$$\|\nabla u\|_p \ge \|\nabla u^t\|_p, \tag{3.19}$$

$$\left\|\frac{\partial u}{\partial x_i}\right\|_p \ge \left\|\frac{\partial u^t}{\partial x_i}\right\|_p, \qquad i = 1, \ldots, n. \tag{3.20}$$

Proof. First observe that if $p \in [1, +\infty)$ and if u is good then (3.19) and (3.20) follow from Corollary 3.1. In the general case we will use various approximation arguments:

(1) Let $1 < p < +\infty$ and $u \in W_+^{1,p}(\mathbb{R}^n)$. We choose a sequence of good functions converging to u in $W^{1,p}(\mathbb{R}^n)$. From the equimeasurability it follows that $u^t, u_m^t \in L^p(\mathbb{R}^n)$, $m = 1, 2, \ldots$, and in view of the nonexpansivity, (Remark 2.6 (5)), we infer that $u_m^t \longrightarrow u^t$ in $L^p(\mathbb{R}^n)$. Further we have that $\|\nabla u_m^t\|_p \le \|\nabla u_m\|_p$ by (3.19), i.e. the functions u_m^t are uniformly bounded. Hence there is a subsequence $(u_{m'})^t$ which converges weakly in $W^{1,p}(\mathbb{R}^n)$ to some $v \in W^{1,p}(\mathbb{R}^n)$. This means that we have for every function $\varphi \in C_0^\infty(\mathbb{R}^n)$ and for every $i \in \{1, \ldots, n\}$

$$-\int_{\mathbb{R}^n} u^t \frac{\partial\varphi}{\partial x_i}\mathrm{d}x \longleftarrow -\int_{\mathbb{R}^n} (u_{m'})^t \frac{\partial\varphi}{\partial x_i}\mathrm{d}x = \int_{\mathbb{R}^n} \varphi \frac{\partial (u_{m'})^t}{\partial x_i}\mathrm{d}x \longrightarrow \int_{\mathbb{R}^n} \varphi \frac{\partial v}{\partial x_i}\mathrm{d}x \quad \text{as } m' \to +\infty,$$

from which we can identify u^t as a function in $W^{1,p}(\mathbb{R}^n)$ with $\nabla u^t = \nabla v$. Since the norm in $W^{1,p}(\mathbb{R}^n)$ is weakly lower semicontinuous, we infer that

$$\|\nabla u^t\|_p \le \lim_{m' \to \infty} \inf \|\nabla (u_{m'})^t\|_p \le \lim_{m' \to \infty} \|\nabla u_{m'}\|_p = \|\nabla u\|_p. \tag{3.21}$$

Further we have $u_{x_i}^t \in L^p(\mathbb{R}^n)$, $i = 1, \ldots, n$. Therefore an estimate for the partial derivatives $u_{x_i}^t$ analogous to (3.21) leads to the inequalities (3.20).

(Note that similar arguments can be found in ([K1], p. 23) and ([BZ], p. 159).)

(2) Let $u \in W^{1,\infty}(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$. We introduce the cut-off functions u_c, $(c \ge 0)$, by

$$u_c := (u - c)_+ = \max\{u - c; 0\}. \tag{3.22}$$

Then

$$|\{u_c > 0\}| < +\infty \qquad \forall c > \inf u. \tag{3.23}$$

It follows that $u_c \in W^{1,p}(\mathbb{R}^n)$ for every $p \in [1, +\infty]$, and in view of $(u_c)^t = (u^t)_c$ we infer that

$$\|\nabla (u^t)_c\|_p \le \|\nabla u_c\|_p \qquad \forall p \in [1, +\infty). \tag{3.24}$$

Because of (3.23) we can pass to the limit $p \to +\infty$ in (3.24) to derive

$$\operatorname{ess\,sup}\{|\nabla (u^t)_c(x)| : x \in \mathbb{R}^n\} \le \operatorname{ess\,sup}\{|\nabla u_c(x)| : x \in \mathbb{R}^n\} \quad \forall c > \inf u.$$

Choosing $c \to 0$ and by taking into account that $\nabla u = 0$ a.e. on $\{u = \inf u\}$ and $\nabla u^t = 0$ a.e. on $\{u^t = \inf u\}$, (3.19) follows in the case $p = +\infty$. Analogous considerations lead to the inequalities (3.20) in the case $p = +\infty$.

(3) Let $u \in W_0^{1,1}(\mathbb{R}^n)$. We choose a sequence of Lipschitz continuous functions with compact support u_m converging to u in $W^{1,1}(\mathbb{R}^n)$. Then we have that $u_m^t \longrightarrow u^t$ in $L^1(\mathbb{R}^n)$. Furthermore, since $\|\partial u_m^t/\partial x_i\|_1 \le \|\partial u_m/\partial x_i\|_1$ by (3.20), we see that the functions $(\partial u_m^t)/(\partial x_i)$ are uniformly bounded in $L^1(\mathbb{R}^n)$, $i = 1, \dots, n$, and for every Young function G we have

$$\int_{\mathbb{R}^n} G\left(\left|\frac{\partial u_m^t(x)}{\partial x_i}\right|\right) \mathrm{d}x \le \int_{\mathbb{R}^n} G\left(\left|\frac{\partial u_m(x)}{\partial x_i}\right|\right) \mathrm{d}x, \qquad m = 1, 2, \dots, \quad i = 1, \dots, n. \tag{3.25}$$

From Lemma 3.2 we infer that for every set $M \in \mathcal{M}(\mathbb{R}^n)$

$$\int_M \left|\frac{\partial u_m^t(x)}{\partial x_i}\right| \mathrm{d}x \le \sup\left\{ \int_N \left|\frac{\partial u_m(x)}{\partial x_i}\right| \mathrm{d}x : |N| \le |M|,\ N \in \mathcal{M}(\mathbb{R}^n) \right\}, \qquad m = 1, 2, \dots, \quad i = 1, \dots, n. \tag{3.26}$$

Now assume for a moment that for every $i \in \{1, \dots, n\}$

$$\sup\left\{ \int_{E_k} \left|\frac{\partial u_m^t(x)}{\partial x_i}\right| \mathrm{d}x : m \in \mathbb{N} \right\} \longrightarrow 0 \qquad \text{as } k \to +\infty, \tag{3.27}$$

for every sequence $\{E_k\} \subset \mathcal{M}(\mathbb{R}^n)$ with $|E_k| \longrightarrow 0$.

From a well known weak compactness principle of sequences in $L^1(\mathbb{R}^n)$, (see e.g. [Alt], p. 199), we infer that there are subsequences $(\partial u_{m'}^t)/(\partial x_i)$ which converge weakly in $L^1(\mathbb{R}^n)$ to functions $v_i \in L^1(\mathbb{R}^n)$, respectively, $i = 1, \dots, n$. By proceeding as in part (1) of the proof one obtains then (3.19) and (3.20). Thus it remains to show (3.27).

Suppose that (3.27) is not true for some $i \in \{1, \dots, n\}$. In view of (3.26) there is a number $\delta > 0$ and sequences $\{m_k\} \subset \mathbb{N}$ and $\{E_k\} \subset \mathcal{M}(\mathbb{R}^n)$ such that $|E_k| \longrightarrow 0$ as $k \to +\infty$ and

$$\sup\left\{ \int_N \left|\frac{\partial u_{m_k}(x)}{\partial x_i}\right| \mathrm{d}x : |N| \le |E_k|, N \in \mathcal{M}(\mathbb{R}^n) \right\} \ge \delta. \tag{3.28}$$

Therefore we can find a sequence $\{N_k\} \subset \mathcal{M}(\mathbb{R}^n)$ with $|N_k| \le |E_k|$, $k = 1, 2, \dots$, such that

$$\int_{N_k} \left|\frac{\partial u_{m_k}(x)}{\partial x_i}\right| \mathrm{d}x \ge \frac{\delta}{2}. \tag{3.29}$$

There are possible two cases:

(a) The sequence $\{m_k\}$ is unbounded. We choose a subsequence $\{k'\}$ with $m_{k'} \to +\infty$ as $k' \to +\infty$. From (3.29) we have that

$$\int_{N_k'} \left|\frac{\partial u(x)}{\partial x_i}\right| \mathrm{d}x \ge \frac{\delta}{4} \qquad \text{for } k' \text{ large enough},$$

which is impossible since $u \in W^{1,1}(\mathbb{R}^n)$.

(b) The sequence $\{m_k\}$ is bounded. Then by passing to a subsequence $\{k'\}$ with $m_{k'} = m$ ($=$ const) in (3.29) we derive a contradiction to $u_m \in W^{1,1}(\mathbb{R}^n)$. ∎

Remark 3.3. There exists a simple alternative proof of Theorem 3.2 by means of an approximation via convolution-type inequalities (see [B4]). Note that this method of proof was developped by Baernstein [Ba] for various types of rearrangements. Unfortunately, this idea seems not applicable in the case of the general inequalities (3.12) and (3.18).

It is easy to obtain an analogue of Theorem 3.2 for functions in the Sobolev spaces $W^{1,p}_{0+}(\Omega)$, (Ω open).

COROLLARY 3.2

Let Ω be an open set and let $u \in W^{1,p}_{0+}(\Omega)$ for some $p \in (1,+\infty)$. Then for every $t \in [0,+\infty]$ we have $u^t \in W^{1,p}_{0+}(\Omega^t)$ and (3.19), (3.20) *hold.*

Proof. Equations (3.19) and (3.20) follow from Theorem 3.2 by extending u and u^t by zero outside Ω and Ω^t, respectively. Thus it remains to show that $u^t \in W^{1,p}_0(\Omega^t)$. If $u \in C^{0,1}_{0+}(\Omega)$ it follows by Remark 2.6 (8) that $u^t \in C^{0,1}_{0+}(\Omega^t)$. In the general case we choose a sequence u_m of functions in $C^{0,1}_{0+}(\Omega)$ which converges to u in $W^{1,p}_0(\Omega)$. Then $(u_m)^t \to u^t$ in $L^p(\Omega^t)$. By (3.19) the functions $(u_m)^t$ are equibounded in $W^{1,p}_0(\Omega^t)$. Therefore there is a function $v \in W^{1,p}_0(\Omega^t)$ and a subsequence $(u_{m'})^t$ which converges to v weakly in $W^{1,p}_0(\Omega^t)$. This means that for every $\varphi \in C^\infty_0(\Omega^t)$ and for every $i \in \{1,\dots,n\}$

$$\int_{\Omega^t} \varphi v_{x_i}\,\mathrm{d}x \longleftarrow \int_{\Omega^t} \varphi \frac{\partial (u_{m'})^t}{\partial x_i}\,\mathrm{d}x = -\int_{\Omega^t} \varphi_{x_i} (u_{m'})^t\,\mathrm{d}x \longrightarrow -\int_{\Omega^t} \varphi_{x_i} u^t \mathrm{d}x \qquad \text{as } m' \to +\infty,$$

that is $v = u^t$. The corollary is proved. ∎

The following property is useful for approximations of the symmetrized functions. It can be proved by arguing as in part (1) of the proof of Theorem 3.2.

Lemma 3.3. *Let $u, u_m \in W^{1,p}_+(\mathbb{R}^n)$, $m = 1,2,\dots$, for some $p \in (1,+\infty)$ and*

$$u_m \longrightarrow u \qquad \text{in } W^{1,p}(\mathbb{R}^n) \qquad \text{as } m \to +\infty. \tag{3.30}$$

Then for every $t \in [0,+\infty]$

$$u^t_m \rightharpoonup u^t \qquad \text{weakly in } W^{1,p}(\mathbb{R}^n) \qquad \text{as } m \to +\infty. \tag{3.31}$$

Open problem 3.1. Let u, u_m be as in Lemma 3.3. Is it then true that for every $t \in [0,+\infty]$

$$u^t_m \longrightarrow u^t \qquad \text{in } W^{1,p}(\mathbb{R}^n) \qquad \text{as } m \to +\infty\,? \tag{3.32}$$

This conjecture was shown in the case $t = +\infty$ (i.e. for the Steiner symmetrization) by Burchard [Bu] (see also [C] for an earlier proof in the particular case that $n = 1$ and $t = +\infty$).

It is worth to mention that, if $n \geq 2$, then a conclusion analogous to (3.32) does not hold for the Schwarz symmetrizations of u, u_m, $m = 1,2,\dots$, (see [AL]).

It is possible to extend Corollary 3.1 to Sobolev functions.

COROLLARY 3.3

Let Ω be an open set with $\Omega = \Omega^*$ and let $u \in W^{1,p}_{0+}(\Omega)$ for some $p \in [1,+\infty)$. Further let G, a, a_{ij}, $i,j = 1,\dots,n-1$, be as in Corollary 3.1, *and suppose that for some numbers*

$C > c > 0$

$$c \leq a(x, v) \leq C, \qquad c \sum_{i=1}^{n-1} \xi_i^2 \leq \sum_{i,j=1}^{n-1} a_{ij}(x', v)\xi_i\xi_j \leq C \sum_{i=1}^{n-1} \xi_i^2$$
$$\forall(\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1} \quad \textit{and} \quad \forall(x, v, z) \in \mathbb{R}^n \times (\mathbb{R}_0^+)^2. \tag{3.33}$$

Finally suppose that

$$G(x', v, z) \leq \begin{cases} Cz^p & \textit{if } |\Omega| = +\infty \\ C(z + z^p) & \textit{if } |\Omega| < +\infty \end{cases}. \tag{3.34}$$

Then (3.18) *holds.*

Proof. We choose a sequence of good functions $\{u_m\}$ such that

$$\begin{aligned} u_m &\longrightarrow u \text{ in } W^{1,p}(\Omega) \quad \text{and} \\ u_m &\longrightarrow u \qquad\qquad\qquad \text{a.e. in } \Omega. \\ \nabla u_m &\longrightarrow \nabla u \end{aligned} \tag{3.35}$$

Let $J(u)$ denote the integral functional on the left-hand side of (3.18). By (3.33) we have

$$J(u_m) \leq \begin{cases} C\|\nabla u_m\|_p^p & \text{if } |\Omega| = +\infty \\ C(\|\nabla u_m\|_1 + \|\nabla u_m\|_p^p) & \text{if } |\Omega| < +\infty \end{cases}, \tag{3.36}$$

and the same inequality holds for u_m replaced by u. In view of (3.34), (3.35) we can apply Lebesgue's convergence theorem to infer that $\lim_{m \to +\infty} J(u_m) = J(u)$.

Let $t \in [0, +\infty]$. Since the functions $(u_m)^t$ are equibounded in $W^{1,p}(\Omega)$ we can choose a subsequence $\{(u_{m'})^t\}$ which converges to u^t weakly in $W^{1,p}(\Omega)$. In view of the weak lower semicontinuity of the functional J this finally gives

$$J(u^t) \leq \liminf_{m' \to +\infty} J((u_{m'})^t) \leq \lim_{m' \to +\infty} J(u_{m'}) = J(u). \qquad \blacksquare$$

Remark 3.4. (1) The inequalities (3.19) find their analogy in inequalities for the norm in the space of functions with bounded variation in $\mathbf{R}^n$ (see [B4]). A consequence of this is that the perimeter of Caccioppoli sets decreases under continuous symmetrization. (2) With regard to some 'nice' properties of the continuous symmetrization – and in particular to the basic fact that the Lipschitz continuity of functions is preserved under continuous rearrangement – our restriction to the Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ is not forcible. The general Dirichlet-type inequality (3.18), for instance, is also satisfied for functions lying in a suitable Orlicz space.

4. Continuity in t

In this section we are interested in continuity properties of the mapping

$$t \longmapsto u^t$$

in the spaces $L^p(\mathbb{R}^n)$, $W^{1,p}(\mathbb{R}^n)$, $1 \leq p < +\infty$, and $BV(\mathbb{R}^n)$.

Lemma 4.1. *Let $u \in L^p_+(\mathbb{R}^n)$ for some $p \in [1, +\infty)$ and let $\{t_m\}$ be a nonnegative sequence converging to some number $t \in [0, +\infty]$. Then*

$$u^{t_m} \longrightarrow u^t \quad \text{in } L^p(\mathbb{R}^n) \quad \text{as } m \to +\infty. \tag{4.1}$$

Proof. The proof is in two steps:

(1) Let u be a step function of the following form,

$$u = \varepsilon \sum_{i=1}^{k} \chi(M_i), \tag{4.2}$$

where $M_1 \supset \cdots \supset M_k$, $M_i \in \mathcal{M}(\mathbb{R}^n)$, $i = 1, \ldots, k$, $\quad \varepsilon > 0$.

Then

$$u^t = \varepsilon \sum_{i=1}^{k} \chi(M_i^t), \quad u^{t_m} = \varepsilon \sum_{i=1}^{k} \chi(M_i^{t_m}), \qquad m = 1, 2, \ldots.$$

In view of (2.14) we have

$$\begin{aligned} \|u^{t_m} - u^t\|_p &= \varepsilon \left\| \sum_{i=1}^{k} (\chi(M_i^{t_m}) - \chi(M_i^t)) \right\|_p \\ &\le \varepsilon \sum_{i=1}^{k} |M_i^{t_m} \Delta M_i^t| \longrightarrow 0 \qquad \text{as } m \to +\infty. \end{aligned}$$

(2) Let $u \in L^p_+(\mathbb{R}^n)$ and $\varepsilon > 0$. We choose a sequence of step functions $\{u_k\}$ which converges to u in $L^p(\mathbb{R}^n)$. We may take k large enough such that $\|u_k - u\|_p < \varepsilon/3$ and then m large enough to ensure that $\|u_k^{|t_m - t|} - u_k\|_p < \varepsilon/3$. In view of the nonexpansivity (2.35) we derive

$$\begin{aligned} \|u^{t_m} - u^t\|_p &\le \|u^{t_m} - u_k^{t_m}\|_p + \|u_k^{t_m} - u_k^t\|_p + \|u_k^t - u^t\|_p \\ &\le 2\|u_k - u\|_p + \|u_k^{|t_m - t|} - u_k\|_p < \varepsilon, \end{aligned}$$

and the assertion follows. ■

Lemma 4.2. *Let $u \in L^p(\mathbb{R}^n)$ and $u \neq u^*$. Then there are constants $c > 0$ and $t_0 > 0$ such that:*

$$\|u^t - u\|_p \ge ct \qquad \forall t \in [0, t_0]. \tag{4.3}$$

Proof. By Lemma 4.1 we can find numbers $t_0 > 0$ and $\delta > 0$ such that $\|u^t - u\|_p \ge \delta$ $\forall t \in [t_0, +\infty]$. If $t \in [0, t_0]$ we find some number $N \in \mathbb{N}$ satisfying $t_0 \le Nt \le 2t_0$. Then by the nonexpansivity we derive

$$\|u^{Nt} - u\|_p \le \sum_{k=0}^{N-1} \|u^{(k+1)t} - u^{kt}\|_p \le N\|u^t - u\|_p,$$

which means that

$$\|u^t - u\|_p \ge \frac{\delta}{N} \ge \frac{\delta}{2t_0} t.$$ ■

Theorem 4.1. *Continuity from the right of the mapping* $t \longmapsto u^t$*: Let* $t_m \searrow 0$ *and* $u \in W_+^{1,p}(\mathbb{R}^n)$ *for some* $p \in [1,+\infty)$*. Then*

$$u^{t_m} \longrightarrow u \qquad \text{in } W^{1,p}(\mathbb{R}^n) \qquad \text{as } m \to +\infty. \tag{4.4}$$

Proof. Let $i \in \{1,\dots,n\}$. We split into two cases:

(1) $p > 1$. From Theorem 3.2 we infer that the sequence $\|u_{x_i}^{t_m}\|_p$ is monotonically increasing and

$$\lim_{m\to\infty} \|u_{x_i}^{t_m}\|_p \le \|u_{x_i}\|_p. \tag{4.5}$$

Furthermore, the sequence $\{u^{t_m}\}$ converges to u in $L^p(\mathbb{R}^n)$ by Lemma 4.1. It follows that for every $\varphi \in C_0^\infty(\mathbb{R}^n)$

$$-\int_{\mathbb{R}^n} \varphi \frac{\partial u^{t_m}}{\partial x_i}\,\mathrm{d}x = \int_{\mathbb{R}^n} u^{t_m}\varphi_{x_i}\,\mathrm{d}x \longrightarrow \int_{\mathbb{R}^n} u\varphi_{x_i}\,\mathrm{d}x = -\int_{\mathbb{R}^n} \varphi u_{x_i}\,\mathrm{d}x,$$

that is

$$\frac{\partial u^{t_m}}{\partial x_i} \rightharpoonup \frac{\partial u}{\partial x_i} \quad \text{weakly in } L^p(\mathbb{R}^n) \quad \text{as } m \to +\infty. \tag{4.6}$$

Since the spaces $L^p(\mathbb{R}^n)$ are uniformly convex if $1 < p < +\infty$, (4.5) and (4.6) imply that

$$\frac{\partial u^{t_m}}{\partial x_i} \longrightarrow \frac{\partial u}{\partial x_i} \qquad \text{strongly in } L^p(\mathbb{R}^n), \qquad i = 1,\dots,n.$$

(2) $p = 1$. As in part (1) we can derive (4.5) and (4.6). We set $v_m := (u^{t_m})_{x_i}$ and $v := (u^t)_{x_i}$. Since the function $G(z) := \sqrt{1+z^2} - 1$ is continuous and convex with $G(z) \le z$ we conclude from Corollary 3.4 and the weak lower semi-continuity of the integral that

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} (\sqrt{1+v_m^2} - 1)\mathrm{d}x = \int_{\mathbb{R}^n} (\sqrt{1+v^2} - 1)\mathrm{d}x.$$

Further we obtain by Taylor's theorem

$$\int_{\mathbb{R}^n} (\sqrt{1+v_m^2} - 1)\mathrm{d}x \ge \int_{\mathbb{R}^n} (\sqrt{1+v^2} - 1)\mathrm{d}x + \int_{\mathbb{R}^n} \frac{v}{\sqrt{1+v^2}}(v_m - v)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} \frac{(v_m - v)^2}{(1+\max\{v^2; v_m^2\})^{3/2}}\mathrm{d}x.$$

By passing to the limit $m \to +\infty$ this leads to

$$\lim_{m\to\infty} \int_{\mathbb{R}^n} \frac{(v_m - v)^2}{(1+\max\{v^2; v_m^2\})^{3/2}}\mathrm{d}x = 0.$$

In particular this means that

$$\lim_{m\to\infty} \int_{\{|v_m|,|v|\le k\}} |v_m - v|\mathrm{d}x = 0 \qquad \forall k > 0. \tag{4.7}$$

Since $v_m \rightharpoonup v$ weakly in $L^p(\mathbb{R}^n)$ we have also

$$\lim_{k\to+\infty} \int_{\{|v_m|>k\}} |v_m|\mathrm{d}x = 0 \qquad \text{uniformly } \forall m \in \mathbb{N}. \tag{4.8}$$

Now the assertion follows easily from (4.7), (4.8) and from the inequalities

$$\|v_m - v\|_1 \le \int_{\{|v_m|,\ |v|\le k\}} |v_m - v|\mathrm{d}x + 2\int_{\{|v_m|>k\}} |v_m|\mathrm{d}x + 2\int_{\{|v|>k\}} |v|\mathrm{d}x \qquad \forall k > 0.$$

■

Remark 4.1. Simple examples of piecewise linear functions show that Theorem 4.1 does not hold in the case $p = +\infty$.

Open problem 4.1. It would be interesting to find out whether the mapping

$$t \longmapsto u^t$$

is also continuous from the left in $W_+^{1,p}(\mathbb{R}^n)$, $1 \le p < +\infty$.

Next we want to estimate $\|u^t - u\|_p$ from above for functions in Sobolev spaces.

Lemma 4.3. *Let u be a good function. Then the limit function*

$$U(x) := \lim_{t\to 0} \frac{1}{t}(u^t(x) - u(x)) \tag{4.9}$$

exists a.e. *Moreover if u is differentiable in* (x', y_i), $i = 1, 2$, *and*

$$y_1 < y_2, \quad u(x', y_1) = u(x', y_2) < u(x', z) \qquad \forall z \in (y_1, y_2), \tag{4.10}$$

then

$$U(x', y_i) = \frac{y_1 + y_2}{2} u_y(x', y_i), \qquad i = 1, 2. \tag{4.11}$$

Proof. For almost every $x^1 = (x', y_1)$ with $u_y(x^1) > 0$ we can find a point $x^2 = (x', y_2)$ such that $u_y(x^2) < 0$ and such that (4.10) is satisfied. We fix two points x^1 and x^2 with these properties. Let $y_i = y_i(x', u)$ be the local inverse function of u in the neighborhood of $x^i, i = 1, 2$, respectively. Then for small enough $t > 0$, the function u^t can be represented by corresponding local inverse functions $y_i^t = y_i^t(x', u)$, $i = 1, 2$, which are given by the formulas (2.10). In other words, we have for small $t > 0$,

$$u^t(x', y_i^t(x', u)) = u(x', y_i), \quad i = 1, 2.$$

Differentiating this we obtain, using (2.10),

$$\left.\frac{\partial u^t(x', y_i)}{\partial t}\right|_{t=0} = -\frac{\partial u^t(x', y_i)}{\partial y} \cdot \left.\frac{\partial y_i^t}{\partial t}\right|_{t=0} = \frac{y_1 + y_2}{2} u_y(x', y_i), \qquad i = 1, 2.$$

Reversely, for almost every $x^2 = (x', y_2)$ with $u_y(x^2) < 0$ we can find a point $x^1 = (x', y_1)$ such that $u_y(x^1) > 0$ and such that (4.10) is satisfied, and we conclude as before. ■

Theorem 4.2. *Let* $u \in W_{0+}^{1,p}(B_R)$ *for some* $p \in [1, +\infty]$ *and* $R > 0$. *Then*

$$\|u^t - u\|_p \le tR\|u_y\|_p \qquad \forall t \in [0, +\infty]. \tag{4.12}$$

Proof. Let u be Lipschitz continuous and let $x_0 = (x_0', y_0) \in B_R$. We set

$$u_1(y) := \max\{0; u(x_0) - \|u_y\|_\infty |y - y_0|\} \qquad \text{and}$$
$$u_2(y) := \max\{0; \min\{u(x_0) + \|u_y\|_\infty |y - y_0|;\ \|u_y\|_\infty (R - |y|)\}\} \qquad \forall y \in \mathbb{R}.$$

Clearly we have $u_1(y) \leq u(x_0) \leq u_2(y)$ and $u_1(y) \leq u(x_0', y) \leq u_2(y)$ $\forall y \in \mathbb{R}$. Let u_i^t denote the (one-dimensional !) continuous symmetrization of the function u_i, $i = 1, 2$, respectively. We obtain by monotonicity

$$u_1^t(y) \leq u(x_0) \leq u_2^t(y) \quad \text{and} \quad u_1^t(y) \leq u^t(x_0', y) \leq u_2^t(y) \quad \forall y \in \mathbb{R}. \tag{4.13}$$

Furthermore, a simple computation shows that

$$\max\{u(x_0) - u_1^t(y_0);\ u_2^t(y_0) - u(x_0)\} \leq tR\|u_y\|_\infty.$$

Together with (4.13) this yields $|u^t(x_0) - u(x_0)| \leq tR\|u_y\|_\infty$, which proves (4.12) in the case $p = +\infty$.

Next let $1 \leq p < +\infty$. First assume that u is a good function. From (4.11) we obtain $|U(x)| \leq R|u_y(x)|$ for a.e. $x \in B_R$. After an integration over B_R this yields

$$\|U\|_p \leq R\|u_y\|_p. \tag{4.14}$$

The functions $(1/t)(u^t - u)$ are equibounded in $L^\infty(B_R)$ by (4.12) and converge to U a.e. in B_R. By applying Lebesgue's convergence theorem we infer that

$$\frac{u^t - u}{t} \longrightarrow U \quad \text{in } L^p(B_R) \qquad \text{as } t \to 0. \tag{4.15}$$

Further we derive from the nonexpansivity

$$\|u^t - u\|_p \leq \sum_{k=0}^{N-1} \|u^{(k+1)t/N} - u^{kt/N}\|_p \leq N\|u^{t/N} - u\|_p \qquad \forall N \in \mathbb{N}.$$

By passing to the limit $N \to +\infty$, this yields $\|u^t - u\|_p \leq t\|U\|_p$ in view of (4.15). Now the assertion follows from (4.14).

In the general case we choose a sequence $\{u_m\}$ of good functions converging to u in $W^{1,p}(B_R)$ and compute

$$\|u^t - u\|_p \leq tR\|(u_m)_y\|_p + \|u_m - u\|_p + \|u_m^t - u^t\|_p \longrightarrow tR\|u_y\|_p \quad \text{as } m \to +\infty.$$

The theorem is proved. ■

Now we prove an analogue of Lemma 4.1 for functions in $C(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$. Note that Lemma 4.5 below generalizes a part of Theorem 7 in [B2].

Lemma 4.4. *Let $u \in C(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$ and $t, \{t_m\}$ as in Lemma* 4.1. *Then*

$$\|u^{t_m} - u^t\|_\infty \longrightarrow 0 \qquad \text{as } m \to +\infty. \tag{4.16}$$

Proof. If $u \in C_{0+}^{0,1}(B_R)$ for some $R > 0$, we obtain from Theorem 4.2 and (4.12) the estimate $\|u^{t_m} - u^t\|_\infty \leq R\ |t_m - t|\|u_y\|_\infty$. In the general case let $\varepsilon > 0$. We choose a nonnegative Lipschitz function v with compact support such that $\|u - v\|_\infty < \varepsilon/3$. By setting $R := \text{diam (supp } v)$ we find some $m_0 \in \mathbb{N}$ such that $|t_m - t| < \varepsilon(3R\|v_y\|_\infty)^{-1}$ $\forall m \geq m_0$. By the nonexpansivity we have for every $m \geq m_0$

$$\begin{aligned} \|u^{t_m} - u^t\|_\infty &\leq \|u^{t_m} - v^{t_m}\|_\infty + \|v^{t_m} - v^t\|_\infty + \|v^t - u^t\|_\infty \\ &\leq 2\|u - v\|_\infty + \|v^{t_m} - v^t\|_\infty < \varepsilon. \end{aligned}$$

The lemma is proved. ■

Formula (4.11) shows that the difference $|u^t(x) - u(x)|$ is proportional to $|x|$. Therefore it is not easy to derive estimates like Theorem 4.2 for functions which do not have bounded support. However this is possible if u satisfies some decaying properties near infinity. We study a typical situation.

Theorem 4.3. *Let $u \in W^{1,p}(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$ for some $p \in [1, +\infty]$. Let $\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be continuous and decreasing, and suppose that φ satisfies*

$$\int_0^{+\infty} r^{(n/p)-1} \varphi(r) \mathrm{d}r < +\infty. \tag{4.17}$$

Further suppose that u satisfies $(\alpha \in (0,1), R, d > 0)$

$$u(x) \geq \varphi(R/\alpha) \qquad \text{if } |x| \leq R \tag{4.18}$$

and

$$\left.\begin{array}{r} \varphi(|x|/\alpha) \leq u(x) \leq \varphi(|x|) \\ |\nabla u(x)| \leq d|x|^{-1}\varphi(|x|) \end{array}\right\} \quad \text{if } |x| \geq R. \tag{4.19}$$

Then there is some constant $C > 0$, depending only on φ, α, R, d and $\|u_y\|_p$, such that

$$\|u^t - u\|_p \leq Ct \qquad \forall t \in [0, +\infty]. \tag{4.20}$$

Furthermore, (4.18) *and the first inequality in* (4.19) *remain valid for u replaced by u^t for every $t \in [0, +\infty]$.*

Proof. Let $t \in [0, +\infty]$. The idea of the proof consists in combining the formulas (4.12) and (4.19) with a 'layer cake' argument.

We introduce the functions

$$u_0 := \max\{u - \varphi(R); 0\} \quad \text{and}$$

$$u_i := \begin{cases} \varphi(2^{i-1}R) - \varphi(2^i R) & \text{if } u > \varphi(2^{i-1}R) \\ u - \varphi(2^i R) & \text{if } \varphi(2^i R) < u \leq \varphi(2^{i-1}R), \quad i = 1, 2, \ldots. \\ 0 & \text{if } u \leq \varphi(2^i R) \end{cases}$$

We have

$$u = \sum_{i=0}^{+\infty} u_i, \tag{4.21}$$

and in view of Definition 2.4

$$u^t = \sum_{i=0}^{+\infty} (u_i)^t. \tag{4.22}$$

From the assumptions we see that supp $u_0 \subset \overline{B_R}$. By applying Theorem 4.2 this leads to

$$\|(u_0)^t - u_0\|_p \leq tR \left\| \frac{\partial u_0}{\partial y} \right\|_p \leq tR \|u_y\|_p. \tag{4.23}$$

Further, if $i \in \mathbb{N}$, we see from (4.19) that supp $u_i \subset \overline{B_{2^i R}}$ and $\nabla u_i(x) = 0$ in $B_{2^{i-1}\alpha R}$. By using (4.12) and (4.18) again we obtain

$$\begin{aligned}\|(u_i)^t - u_i\|_p &\le t\,2^iR\left\|\frac{\partial u_i}{\partial y}\right\|_p \le t\,2^iR\|\nabla u_i\|_\infty |\text{supp}|\nabla u_i||^{1/p}\\ &\le t\,2^iRd(2^{i-1}\alpha R)^{-1}\varphi(2^{i-1}\alpha R)(\omega_n)^{1/p}((2^iR)^n - (2^{i-1}\alpha R)^n)^{1/p}\\ &\le t\,(\omega_n)^{1/p}d\alpha^{-1}(2^iR)^{n/p}\varphi(2^{i-1}\alpha R) \equiv t\,C_i. \end{aligned} \tag{4.24}$$

In view of (4.17) we derive easily

$$\sum_{i=1}^{+\infty} C_i < +\infty. \tag{4.25}$$

Finally, by using (4.21)–(4.25) we obtain

$$\|u^t - u\|_p \le \sum_{i=0}^{+\infty} \|(u_i)^t - u_i\|_p \le t\left(R\|u_y\|_p + \sum_{i=1}^{+\infty} C_i\right) \equiv tC,$$

and (4.20) follows. Now set

$$\varphi_1(x) := \begin{cases}\varphi(|x|/\alpha) & \text{if } |x| > R\\ \varphi(R/\alpha) & \text{if } |x| \le R\end{cases} \quad \text{and}$$
$$\varphi_2(x) := \begin{cases}\varphi(|x|) & \text{if } |x| > R\\ +\infty & \text{if } |x| \le R\end{cases}.$$

We have $\varphi_1 \le u \le \varphi_2$, by (4.18), (4.19), and clearly $\varphi_i = (\varphi_i)^\star$, $i = 1, 2$. By the monotonicity of continuous symmetrization this means that $\varphi_1 = (\varphi_1)^t \le u^t \le (\varphi_2)^t = \varphi_2$ $\forall t \in [0, +\infty]$, which proves the second assertion of the lemma. ∎

Remark 4.2. It is easy to verify that the function u in the cases (1) and (2) below satisfies the assumptions of Theorem 4.4 ($R > 1, \delta, c_1, c_2, c_3, \gamma, \lambda > 0, \sigma, \tau \in \mathbb{R}$).

(1) $u \in W_+^{1,p}(\mathbb{R}^n)$ for some $p \in [1, +\infty]$ and u satisfies

$$u(x) \ge \delta \quad \text{if } |x| \le R \tag{4.26}$$

and one of the following conditions (i) or (ii)

(i) $$\gamma > (n/p) \tag{4.27}$$

and

$$\left.\begin{aligned} c_1|x|^{-\gamma}(\log|x|)^{-\sigma} \le u(x) \le c_2|x|^{-\gamma}(\log|x|)^{-\sigma}\\ |\nabla u(x)| \le c_3|x|^{-\gamma-1}(\log|x|))^{-\sigma}\end{aligned}\right\} \quad \text{if } |x| \ge R,$$

(ii) $$\left.\begin{aligned} c_1e^{-\lambda|x|}|x|^\tau \le u(x) \le c_2e^{-\lambda|x|}|x|^\tau\\ |\nabla u(x)| \le c_3e^{-\lambda|x|}|x|^\tau\end{aligned}\right\} \quad \text{if } |x| \ge R. \tag{4.28}$$

(2) $u \in W^{1,\infty}(\mathbb{R}^n) \cap \mathcal{S}_+(\mathbb{R}^n)$, $\sigma > 0$ and u satisfies (4.26) and

$$\left.\begin{aligned} (\log(c_1|x|))^{-\sigma} \le u(x) \le (\log(c_2|x|))^{-\sigma}\\ |\nabla u(x)| \le c_3|x|^{-1}(\log(c_2|x|))^{-\sigma-1}\end{aligned}\right\} \quad \text{if } |x| \ge R. \tag{4.29}$$

5. More estimates

In this section we intend to show estimates of the form

$$\liminf_{t\searrow 0} \frac{1}{t}\int_\Omega f(x,u)(u^t-u)\mathrm{d}x \geq 0, \qquad u\in W_+^{1,p}(\Omega), \tag{5.1}$$

for suitable functions f and domains Ω in $\mathbb{R}^n$. At this stage it is worth to sketch a proof of (5.1) in a special case.

Let $f=f(u)$ be continuous and $F(u)=\int_0^u f(v)\mathrm{d}v$, and let Ω be a bounded domain with $\Omega=\Omega^*$. The equimeasurability of u and u^t yields $\int F(u)=\int F(u^t)$. Furthermore, $f(u)\,(u^t-u)$ represents the first summand in the (formal) Taylor expansion of the difference $F(u^t)-F(u)$ into powers of (u^t-u). Heuristically this means that $\int f(u)(u^t-u)$ is small in $\|u^t-u\|_p$. Applying Lemma 4.2 and Theorem 4.2 this finally gives

$$\int_\Omega f(u)(u^t-u)\mathrm{d}x = o(t) \quad \text{as } t\searrow 0.$$

The next Theorem 5.1 can be seen in a certain sense as a generalization of the Hardy–Littlewood inequality (2.36).

Theorem 5.1. *Let Ω be an open set with $\Omega=\Omega^*$. Let $u\in L_+^p(\mathbb{R}^n)$ for some $p\in[1,+\infty)$, and suppose that u vanishes outside Ω. Furthermore, let $F=F(x,v)$ measurable on $\Omega\times[0,\sup u]$, continuous in v and satisfies*

$$F(x,0)=0 \qquad \forall x\in\Omega,$$
$$|F(x,v)|\leq A(x)B(x',v) \qquad \forall (x,v)\in\Omega\times[0,\sup u], \tag{5.2}$$

where $B(x',v)$ is nonnegative, measurable in x' and nondecreasing and continuous in v, and

$$A\in L_+^{1/(1-\alpha)}(\Omega), B(\cdot,u(\cdot))\in L^{1/\alpha}(\Omega)$$

for some $\alpha\in[0,1]$.

Finally, suppose that for every $s>0$ the function

$$\varphi_s(x,v) := F(x,v+s)-F(x,v) \tag{5.3}$$

is symmetrically nonincreasing in y. Then

$$\int_\Omega F(x,u)\mathrm{d}x \leq \int_\Omega F(x,u^t)\mathrm{d}x. \tag{5.4}$$

Remark 5.1. If F is differentiable in v, then the condition (5.3) means that $(\partial F/\partial v)(x,v)$ is symmetrically nonincreasing in y.

Proof of Theorem 5.1. The proof is in two steps:

(1) First assume that u is a step function of the form (4.2). We compute

$$\int_\Omega F(x,u)\mathrm{d}x = \sum_{i=1}^k \int_{M_i}(F(x,\varepsilon i)-F(x,\varepsilon(i-1)))\mathrm{d}x \quad \text{and}$$
$$\int_\Omega F(x,u^t)\mathrm{d}x = \sum_{i=1}^k \int_{M_i^t}(F(x,\varepsilon i)-F(x,\varepsilon(i-1)))\mathrm{d}x. \tag{5.5}$$

Note that the integrals on the right-hand sides in (5.5) converge by (5.2). By (5.3) the functions $\varphi_i(x) := F(x, \varepsilon i) - F(x, \varepsilon(i-1))$ are symmetrically nonincreasing in y. We claim that

$$\int_{M_i} \varphi_i(x)\mathrm{d}x \le \int_{M_i^t} \varphi_i(x)\mathrm{d}x, \quad i = 1, \ldots, k. \tag{5.6}$$

To this end it is sufficient to prove that

$$\int_{M_i(x')} \varphi_i(x', y)\mathrm{d}y \le \int_{(M_i(x'))^t} \varphi_i(x', y)\mathrm{d}y \qquad \text{for a.e. } x' \in \mathbb{R}^{n-1}, \quad i = 1, \ldots, k. \tag{5.7}$$

We fix $x' \in \mathbb{R}^{n-1}$ and $i \in \{1, \ldots, k\}$ such that both integrals in (5.7) converge. Assume first that $\varphi(y) := \varphi_i(x', y)$ is a step function of the form

$$\varphi = \delta\left(-C + \sum_{j=1}^{l} \chi(N_j)\right), \tag{5.8}$$

where $N_j \in \mathcal{M}(\mathbb{R})$, $N_j = N_j^*$, $j = 1, \ldots, l$, $N_1 \supset \cdots \supset N_l$, $C \ge 0$, $\delta > 0$.

By the monotonicity (2.13) we have that

$$\begin{aligned} \int_{M_i(x')} \varphi(y)\mathrm{d}y &= \delta\left(-C|M_i(x')| + \sum_{j=1}^{l} |M_i(x') \cap N_j|\right) \\ &\le \delta\left(-C|(M_i(x'))^t| + \sum_{j=1}^{l} |(M_i(x'))^t \cap N_j|\right) \\ &= \int_{(M_i(x'))^t} \varphi(y)\mathrm{d}y. \end{aligned}$$

A general φ can be approximated in $L^p(\mathbb{R})$ by step functions of the form (5.8). This proves (5.7) and thus (5.6) is established. Now in view of (5.5) we obtain (5.4).

(2) Next let $u \in L^p(\Omega)$. In view of (5.2) both integrals in (5.4) converge. We can choose an increasing sequence of step functions $\{u_m\}$ of the form (4.2) which converges to u in $L^p(\Omega)$. Then we have by (5.2)

$$\begin{aligned} &|F(x, u_m(x))| \le A(x)B(x', u_m(x)) \le A(x)B(x', u(x)) =: f(x), \\ &|F(x, u_m^t(x))| \le A(x)B(x', u_m^t(x)) \le A(x)B(x', u^t(x)) =: g(x) \qquad \forall x \in \Omega \\ &\qquad \text{and} \quad f, g \in L^1(\Omega). \end{aligned} \tag{5.9}$$

By Lemma 4.1 we can choose a subsequence $\{u_{m'}\}$ such that

$$\begin{matrix} u_{m'}(x) \longrightarrow u(x) \\ u_{m'}^t(x) \longrightarrow u^t(x) \end{matrix} \qquad \text{for a.e. } x \in \Omega. \tag{5.10}$$

In view of (5.9) and (5.10) and since $F(x, v)$ is continuous in v, (5.4) follows by Lebesgue's convergence theorem. ■

Theorem 5.2. *Let Ω be an open set with $\Omega = \Omega^*$. Let $u \in L_+^p(\mathbb{R}^n)$ for some $p \in [1, +\infty)$ and suppose that u vanishes outside Ω and satisfies* (4.20). *Furthermore, suppose that*

$f = f(x, v)$ is measurable on $\Omega \times [0, \sup u]$, symmetrically nonincreasing in y and ($p^{-1} + q^{-1} = 1$)

$$|f(x, v)| \leq a(x)b(x', v) \qquad \forall (x, v) \in \Omega \times [0, \sup u], \tag{5.11}$$

where $b(x', v)$ is nonnegative, measurable in x' and nondecreasing and right-continuous in v, and

$$a \in L_+^{q/(1-\beta)}(\Omega), b(\cdot, u(\cdot)) \in L^{q/\beta}(\Omega)$$

for some $\beta \in [0, 1]$. Finally, assume that u,f satisfy one of the conditions (i)–(iv):

(i) *$f(x, v)$ is nonincreasing in v;*
(ii) *$f(x, v)$ satisfies a Hölder condition in v with exponent $\lambda \in [1 - p^{-1}, 1]$, uniformly for every $x \in \Omega$, and $u \in L^{\lambda q}(\Omega)$;*
(iii) *$f(x, v)$ is continuous in v and bounded;*
(iv) *$f(x, v) = h(x)k(x', v)$, where $h \in L_+^{q/(1-\beta)}(\Omega)$, h is symmetrically nonincreasing in y, $k(x', v)$ is nonnegative, measurable in x' and nondecreasing in v and $k(\cdot, u(\cdot)) \in L^{q/\beta}(\Omega)$.*

Then (5.1) *holds, and in case* (i) *we have*

$$\int_\Omega f(x, u)(u^t - u)\mathrm{d}x \geq 0 \qquad \forall t \in [0, +\infty]. \tag{5.12}$$

Proof. We set

$$F(x, v) := \int_0^v f(x, w)\mathrm{d}w \qquad \forall (x, v) \in \Omega \times [0, \sup u]. \tag{5.13}$$

Since

$$|F(x, v)| \leq a(x) \int_0^v b(x', w)\mathrm{d}w \leq a(x)vb(x', v) \qquad \forall (x, v) \in \Omega \times [0, \sup u],$$

we see that F satisfies the assumptions of Theorem 5.1. Thus we derive

$$0 \leq \int_\Omega (F(x, u^t) - F(x, u))\mathrm{d}x = \int_0^1 \int_\Omega f(x, u + \theta(u^t - u))(u^t - u)\mathrm{d}x\,\mathrm{d}\theta.$$

This means that

$$\begin{aligned}\int_\Omega f(x, u)(u^t - u)\mathrm{d}x &\geq \int_0^1 \int_\Omega (f(x, u) - f(x, u + \theta(u^t - u)))(u^t - u)\mathrm{d}x\,\mathrm{d}\theta \\ &=: I(t).\end{aligned} \tag{5.14}$$

Note that in view of the assumptions on f the integral $I(t)$ converges. Now in case (i) we immediately derive (5.12) from (5.14).

Furthermore, we obtain by Hölder's inequality and by (4.20)

$$\begin{aligned}|I(t)| &\leq \|u^t - u\|_p \int_0^1 \|f(\cdot, u + \theta(u^t - u)) - f(\cdot, u)\|_q \mathrm{d}\theta \\ &\leq Ct \int_0^1 \|f(\cdot, u + \theta(u^t - u)) - f(\cdot, u)\|_q \mathrm{d}\theta.\end{aligned} \tag{5.15}$$

In view of (5.14) and (5.15) it suffices to prove that

$$\int_0^1 \|f(\cdot, u+\theta(u^t-u)) - f(\cdot, u)\|_q \mathrm{d}\theta \longrightarrow 0 \qquad \text{as } t \to 0. \tag{5.16}$$

In the case (ii) we obtain

$$\|f(\cdot, u+\theta(u^t-u)) - f(\cdot, u)\|_q^q \le C\|u^t-u\|_{\lambda q}^{\lambda q},$$

and (5.16) follows.

Next consider case (iii), and assume that (5.16) is not true. Then there is a sequence $t_m \searrow 0$ such that

$$\int_0^1 \|f(\cdot, u+\theta(u^{t_m}-u)) - f(\cdot, u)\|_q \mathrm{d}\theta \ge \delta \tag{5.17}$$

for some $\delta > 0$. By passing to a subsequence $\{t_{m'}\}$ we can achieve that $u^{t_{m'}}(x) \longrightarrow u(x)$ a.e. in Ω as $m' \to +\infty$. This yields

$$f(x, u(x)+\theta(u^{t_{m'}}(x)-u(x))) \longrightarrow f(x,u(x)) \qquad \text{a.e. in } \Omega,$$
$$\text{as } m' \to +\infty, \qquad \theta \in [0,1]. \tag{5.18}$$

If Ω is bounded, then by Lebesgue's convergence theorem (5.17) immediately yields a contradiction. If Ω is unbounded, then we derive by Lebesgue's theorem

$$\int_0^1 \|f(\cdot, u+\theta(u^{t_{m'}}-u)) - f(\cdot, u)\|_{q,[\Omega\cap B_R]} \mathrm{d}\theta \longrightarrow 0 \quad \text{as } m' \to \infty \quad \forall R > 0. \tag{5.19}$$

Furthermore, from the Hardy–Littlewood inequality (2.36) we have for every $R > 0$

$$\begin{aligned}\|b(\cdot, u(\cdot))\|_{q/\beta,[\Omega\cap B_R]}^{q/\beta} &= \int_\Omega b(x', u)^{q/\beta}\chi_{B_R}\mathrm{d}x \le \int_\Omega b(x', u^t)^{q/\beta}\chi_{B_R}\mathrm{d}x \\ &= \|b(\cdot, u^t(\cdot))\|_{q/\beta,[\Omega\cap B_R]}^{q/\beta}, \qquad t \in [0, +\infty].\end{aligned}$$

Together with the assumptions (5.11) this yields

$$\begin{aligned}&\int_0^1 \|f(\cdot, u+\theta(u^t-u)) - f(\cdot, u)\|_{q,[\Omega\setminus B_R]} \mathrm{d}\theta \\ &\le \|a\|_{q/(1-\beta),[\Omega\setminus B_R]}(\|b(\cdot, u^{t_{m'}}(\cdot))\|_{q/\beta,[\Omega\setminus B_R]} + \|b(\cdot, u(\cdot))\|_{p,[\Omega\setminus B_R]}) \\ &\le 2\|a\|_{q/(1-\beta),[\Omega\setminus B_R]}\|b(\cdot, u(\cdot))\|_{q/\beta,[\Omega\setminus B_R]} \\ &\longrightarrow 0 \quad \text{as } R \to \infty, \text{ uniformly } \forall m' \in \mathbb{N}.\end{aligned} \tag{5.20}$$

Now (5.20) together with (5.19) contradict to (5.16). Finally, in the case (iv) we have $k(\cdot, u(\cdot)) \in L^{q/\beta}(\Omega)$ and from Definition 2.4 we infer

$$k(\cdot, u^t(\cdot)) = (k(\cdot, u(\cdot)))^t.$$

By using Lemma 4.1 this yields

$$\int_0^1 \|f(\cdot, u+\theta(u^t-u)) - f(u)\|_q \mathrm{d}\theta$$

$$\leq \|h\|_{q/(1-\beta)} \int_0^1 \|k(\cdot, u + \theta(u^t - u)) - k(\cdot, u)\|_{q/\beta} \, \mathrm{d}\theta$$
$$\leq \|h\|_{q/(1-\beta)} \|(k(\cdot, u))^t - k(\cdot, u)\|_{q/\beta} \longrightarrow 0 \qquad \text{as } t \to 0.$$

The Theorem is proved. ■

Remark 5.2. (1) The conditions (5.2) and (5.11) ensure in essence the applicability of Lebesgue's convergence theorem in the proofs, and we may replace these conditions by similar other ones. (2) Note that, if u is bounded in Theorem 5.2, then (ii) is a special case of (iii) by (5.11). Thus the case (ii) is meaningful only for unbounded functions u.

The proof of Theorem 5.2 is based on an estimate for the function u of the form (4.20). If $\Omega = \mathbb{R}^n$, then such an estimate can be ensured if $u \in W_+^{1,p}(\mathbb{R}^n)$ and if u has some decaying properties near infinity (see Theorem 4.4 and Remark 4.2). On the other hand, these estimates could be rather restrictive for some applications. Fortunately under an additional (weak) assumption on $f(x, v)$ we can bypass any strong decaying requirement for u.

Lemma 5.1. *Let $u \in W_+^{1,p}(\mathbb{R}^n) \cap C(\mathbb{R}^n)$ for some $p \in [1, +\infty]$, let $u > 0$ in $\mathbb{R}^n$ and*

$$u(x) \longrightarrow 0 \qquad \text{as } |x| \to +\infty. \tag{5.21}$$

Furthermore, let f together with u satisfy the assumptions of Theorem 5.2 *with Ω replaced by $\mathbb{R}^n$ and, in addition, suppose that for some numbers $R, \delta > 0$*

$$f(x, v) \quad \textit{is nonincreasing in } v \textit{ for } 0 < v < \delta \textit{ and } |x| > R. \tag{5.22}$$

Then the conclusions of Theorem 5.2 *hold.*

Proof. We proceed similarly as in the previous proof. First we obtain (5.14) with $\Omega = \mathbb{R}^n$. If $f(x, v)$ is nonincreasing in v, then we can argue exactly as before. In the remaining cases (ii)–(iv) we choose R_0 large enough and $R_0 > R$ such that $u \leq \delta$ in $\mathbb{R}^n \setminus B_{R_0}$. Then it follows by monotonicity that $u^t \leq \delta$ in $\mathbb{R}^n \setminus B_{R_0}$ for every $t \in [0, +\infty]$. Thus we have

$$I(t) \geq \int_0^1 \int_{B_{R_0}} (f(x, u) - f(x, u + \theta(u^t - u)))(u^t - u) \mathrm{d}x \, \mathrm{d}\theta =: I_1(t). \tag{5.23}$$

We choose $\varepsilon > 0$ small enough such that $u \geq \varepsilon$ in B_{R_0}. Again we have $u^t \geq \varepsilon$ in B_{R_0} for every $t \in [0, +\infty]$ by monotonicity. By setting $u_\varepsilon := \max\{u; \varepsilon\}$, we see that

$$u = u_\varepsilon, \; u^t = (u_\varepsilon)^t \quad \text{in } B_{R_0} \quad \forall t \in [0, +\infty]. \tag{5.24}$$

In view of (5.21) $(u_\varepsilon - \varepsilon)$ has bounded support, i.e. $(u_\varepsilon - \varepsilon) \in W_{0+}^{1,p}(B_{R_1})$ for some $R_1 > R_0$. An application of Theorem 4.2 to u_ε yields

$$\|(u_\varepsilon)^t - u_\varepsilon\|_p \leq tR_1 \|(u_\varepsilon)_y\|_p \leq tR_1 \|u_y\|_p \qquad \forall t \in [0, +\infty]. \tag{5.25}$$

Now by using (5.23)–(5.25) we compute finally

$$|I_1(t)| \leq \|u^t - u\|_{p, B_{R_0}} \int_0^1 \|f(\cdot, u + \theta(u^t - u)) - f(\cdot, u)\|_{q, B_{R_0}} \mathrm{d}\theta$$
$$= \|(u_\varepsilon)^t - u_\varepsilon\|_{p, B_{R_0}} \int_0^1 \|f(\cdot, u_\varepsilon + \theta((u_\varepsilon)^t - u_\varepsilon)) - f(\cdot, u_\varepsilon)\|_{q, B_{R_0}} \mathrm{d}\theta$$

$$\leq \|(u_\varepsilon)^t - u_\varepsilon\|_p \int_0^1 \|f(\cdot, u_\varepsilon + \theta((u_\varepsilon)^t - u_\varepsilon)) - f(\cdot, u_\varepsilon)\|_q \mathrm{d}\theta$$
$$\leq tR_1\|u_y\|_p \int_0^1 \|f(\cdot, u_\varepsilon + \theta((u_\varepsilon)^t - u_\varepsilon)) - f(\cdot, u_\varepsilon)\|_q \mathrm{d}\theta,$$

and the assertions follow by proceeding as in the previous proof. ■

For some applications it will be useful to have an estimate like (5.1) with the function f replaced by *any* element of the (set-valued) *maximal monotone graph* $\tilde{f}(x, v)$ of f with respect to v (compare Remark 7.1(4)). $\tilde{f}$ is defined by

$$\tilde{f}(x, v) := \left[\liminf_{h\to 0} f(x, v+h), \limsup_{h\to 0} f(x, v+h)\right] \quad \forall (x, v) \in \Omega \times [0, \sup]. \tag{5.26}$$

Note that, if F is defined by (5.13), then we can write alternatively

$$\tilde{f}(x, v) = \partial_v F(x, v),$$

where $\partial_v F$ is the set-valued differential of $F(x, v)$ with respect to v,

$$\partial_v F(x, v) := \left[\liminf_{h\to 0} \frac{F(x, v+h) - F(x, v)}{h},\ \limsup_{h\to 0} \frac{F(x, v+h) - F(x, v)}{h}\right]$$
$$\forall (x, v) \in \Omega \times [0, \sup u]. \tag{5.27}$$

COROLLARY 5.1

The conclusions of Theorem 5.2 *and Lemma* 5.1 *hold if the function* $f(\cdot, u(\cdot))$ *in* (5.1) *is replaced by any function* g *with* $g(\cdot) \in \tilde{f}(\cdot, u(\cdot))$.

Proof. From (5.11) we obtain that $|g(x)| \leq a(x)b(x', u(x))$ $\forall x \in \Omega$, which means that $g \in L^q(\Omega)$. This ensures the convergence of the integral $\int_\Omega g(x)(u^t - u)\mathrm{d}x$. Obviously, there is nothing to prove if $f(x, v)$ is continuous in v. In the remaining cases we proceed as in the proof of Theorem 5.2 to infer that

$$\int_\Omega g(x)(u^t - u)\mathrm{d}x \geq \int_0^1 \int_\Omega (g(x) - f(x, u + \theta(u^t - u)))(u^t - u)\mathrm{d}x\, \mathrm{d}\theta$$
$$=: I_2(t). \tag{5.28}$$

If $f(x, v)$ is nonincreasing v, then we have

$$(g(x) - f(x, u(x) + \theta(u^t(x) - u(x))))(u^t(x) - u(x)) \geq 0 \quad \forall x \in \Omega, \quad \theta \in [0, 1].$$

Together with (5.28) this leads to

$$\int_\Omega g(x)(u^t - u)\mathrm{d}x \geq 0 \qquad \forall t \in [0, +\infty]. \tag{5.29}$$

Furthermore, if $f(x, z)$ is nondecreasing in z, then

$$|g(x) - f(x, u(x) + \theta(u^t(x) - u(x)))||u^t(x) - u(x)|$$
$$\leq |f(x, u^t(x)) - f(x, u(x))||u^t(x) - u(x)| \qquad \forall x \in \Omega, \quad \theta \in [0, 1].$$

In view of (5.28) we infer that

$$|I_2(t)| \le \|u^t - u\|_p \|f(\cdot, u^t) - f(\cdot, u)\|_q.$$

Then the assertion follows by proceeding as in the proof of Theorem 5.2. ■

Remark 5.3. The reader verifies easily that Theorem 5.2, Lemma 5.1 and Corollary 5.1 hold true if the function f can be decomposed into a finite sum $\sum_{i=1}^{k} f_i$ where each of the functions f_i, $i = 1, \ldots, k$, satisfies at least one of the conditions (i)–(iv) of Theorem 5.2.

6. Local symmetry

In this section we study functions satisfying the 'local' symmetry property (LS) from the Introduction and the relation to continuous symmetrization.

DEFINITION 6.1 (Local symmetry)

Let $u \in \mathcal{S}_+(\mathbb{R}^n)$ and continuously differentiable on $\{x : 0 < u(x) < \sup u\}$, and suppose that this last set is open. Further, suppose that u has the following property. If $x^1 = (x_0', y_1) \in \mathbb{R}^n$ with

$$0 < u(x^1) < \sup u, \ \frac{\partial u}{\partial y}(x^1) > 0, \tag{6.1}$$

and x^2 is the (unique!) point satisfying

$$x^2 = (x_0', y_2), \quad y_2 > y_1, \quad u(x_1) = u(x^2) < u(x_0', y) \quad \forall y \in (y_1, y_2), \tag{6.2}$$

then

$$\frac{\partial u}{\partial x_i}(x^1) = \frac{\partial u}{\partial x_i}(x^2), \quad i = 1, \ldots, n-1, \quad \text{and}$$
$$\frac{\partial u}{\partial y}(x^1) = -\frac{\partial u}{\partial y}(x^2). \tag{6.3}$$

Then u is called locally symmetric in the direction y.

Remark 6.1. *Geometrical meaning of local symmetry*: (1) The condition $u \in \mathcal{S}_+(\mathbb{R}^n) \cap C^1(\{0 < u < \sup u\})$ is satisfied in the following typical cases (a) and (b).

(a) $u \in C^1_+(\mathbb{R}^n)$ and $\lim_{|x| \to \infty} u(x) = 0$.
(b) There are two bounded open sets $\Omega_i \subset \mathbb{R}^n$, $i = 0, 1$, with $\Omega_0 \subset\subset \Omega_1$, $u \in C(\mathbb{R}^n \setminus \overline{\Omega_0}) \cap C^1(\Omega_1 \setminus \overline{\Omega_0})$, $0 < u < \sup u$ in $\Omega_1 \setminus \overline{\Omega_0}$, $u \equiv 0$ in $\mathbb{R}^n \setminus \overline{\Omega_1}$, $u \equiv \sup u$ in Ω_0 and

$$u(x) \longrightarrow \sup u \quad \text{if } x \to \partial\Omega_0, \quad x \in \Omega_1 \setminus \overline{\Omega_0}.$$

Note that we did not exclude $\sup u = +\infty$ in case (b). (2) Let u, x^1, x^2 be as in Definition 6.1 and let U_1 be the maximal connected component of $\{0 < u < \sup u\} \cap \{u_y > 0\}$ containing x^1. Since $u \in C^1(\{0 < u < \sup u\})$ we have that for every $(x', y) \in U_1$

$$u(x', y) = u(x', y_1 + y_2 - y) < u(x', z) \qquad \forall z \in (y, y_1 + y_2 - y). \tag{6.4}$$

The condition (6.4) says that U_1 finds a congruent counterpart after reflection about some hyperplane $\{y = \text{const}\}$. Repeating this consideration for arbitrary components of

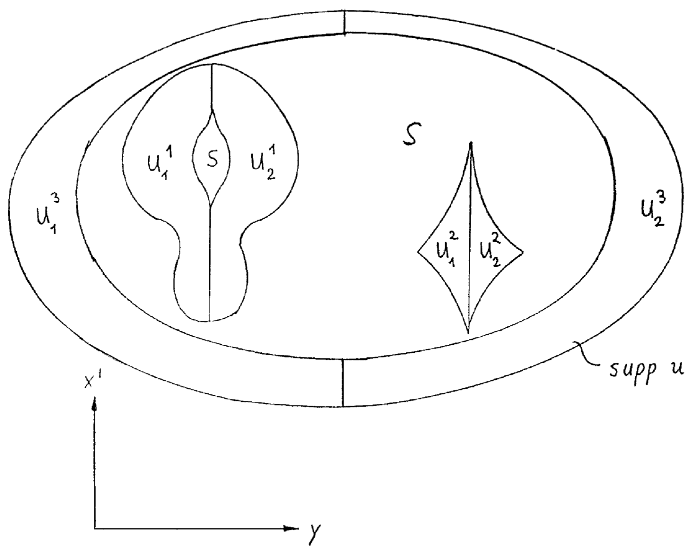

Figure 4.

$\{0 < u < \sup u\} \cap \{u_y > 0\}$ we infer the decomposition

$$\{0 < u < \sup u\} = \bigcup_{k=1}^{m} (U_1^k \cup U_2^k) \bigcup S. \tag{6.5}$$

Here U_1^k is some maximal connected component of $\{0 < u < \sup u\} \cap \{u_y > 0\}$, U_2^k is its reflection about some hyperplane $\{y = d_k\}$, $d_k \in \mathbb{R}$, and we have

$$u_y = 0 \qquad \text{in } S, \tag{6.6}$$

and for every $(x', y) \in U_1^k$,

$$u(x', y) = u(x', 2d_k - y) < u(x', z) \quad \forall z \in (y, 2d_k - y), \quad k = 1, \ldots, m. \tag{6.7}$$

Note that all the sets on the right-hand side of (6.5) are disjoint and there can be a countable number of U_1^k's, i.e. $m = +\infty$.

In many applications we need an 'isotropic' variant of local symmetry.

DEFINITION 6.2

Let u be as in Definition 6.1. u is called locally symmetric in every direction if for every rotation of the cartesian coordinate system $x \longmapsto \xi = (\xi', \eta)$, $\xi' \in \mathbb{R}^{n-1}$, $\eta \in \mathbb{R}$, the function $v(\xi) := u(x)$ is locally symmetric with respect to η.

Surprisingly it can be proved that functions which are locally symmetric in *every* direction are 'locally' *radially symmetric* (see figure 5).

Theorem 6.1. *Let u be locally symmetric in every direction. Then we have the following decomposition,*

$$\{0 < u < \sup u\} = \bigcup_{k=1}^{m} A_k \bigcup S, \tag{6.8}$$

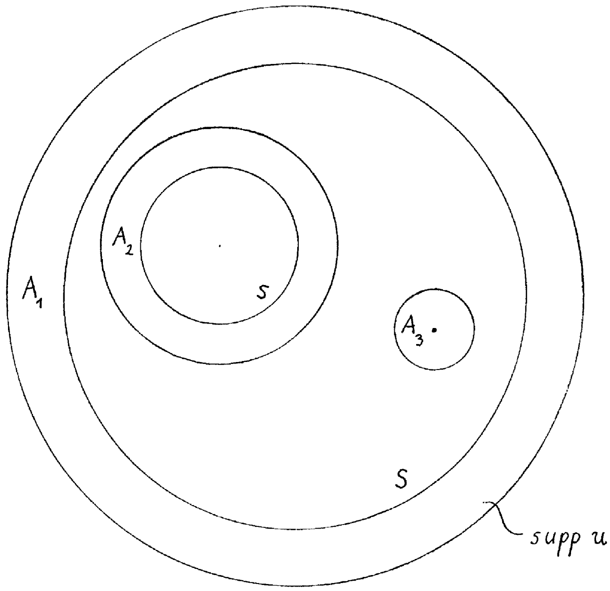

Figure 5.

where the A_k's are pairwise disjoint annuli $B_{R_k}(z_k) \setminus \overline{B_{r_k}(z_k)}$ with $R_k > r_k \geq 0$, $z_k \in \{0 < u < \sup u\}$, u is radially symmetric in A_k, and more precisely,

$$u = u(|x - z_k|), \frac{\partial u}{\partial \rho} < 0 \quad \text{in } A_k, \tag{6.9}$$

$$(\rho = |x - z_k|), \quad \text{and}$$

$$u(x) \geq u|_{\partial B_{r_k}(z_k)} \quad \forall x \in B_{r_k}(z_k),$$

$$1 \leq k \leq m. \tag{6.10}$$

Furthermore, we have

$$\nabla u = 0 \qquad \text{in } S, \tag{6.11}$$

and there can be a countable number of annuli, i.e. $m = +\infty$. Finally, if $\{u > 0\}$ is unbounded, then the case $R_1 = +\infty$ is possible.

Proof. We use the notations of Definition 6.1. Let x^1, x^2 be two points which satisfy (6.1), (6.2), let U_1 be the connected component of $\{0 < u < \sup u\} \cap \{x : u_y(x) > 0\}$ containing x^1 and suppose that $\overline{x}$ is some point in U_1 with $u(x^1) = u(\overline{x})$. By a suitable rotation of the coordinate system $x \longmapsto \xi = (\xi', \eta)$, $\xi' \in \mathbb{R}^{n-1}, \eta \in \mathbb{R}$, about the point x^2 we can achieve that the ray connecting $\overline{x}$ and x^2 points into the positive η-direction, i.e. $\overline{x} \longmapsto \xi^1 = (\xi_0', \eta_1)$ and $x^2 \longmapsto \xi^2 = (\xi_0', \eta_2)$. We set $v(\xi) := u(x)$. It is easy to see that, if the distance $|x^1 - \overline{x}|$ is small enough (say $|x^1 - \overline{x}| < \varepsilon$), then $v_\eta(\xi^1) > 0$ and $v(\xi^1) = v(\xi^2) < v(\xi_0', \eta)$ $\forall \eta \in (\eta_1, \eta_2)$. By the assumptions this means that

$$\frac{\partial v}{\partial \eta}(\xi^1) = -\frac{\partial v}{\partial \eta}(\xi^2) > 0 \qquad \text{and}$$

$$\frac{\partial v}{\partial \xi_i}(\xi^1) = \frac{\partial v}{\partial \xi_i}(\xi^2), \quad i = 1, \ldots, n-1.$$

From a simple computation follows that the set

$$\Gamma_1 := U_1 \cap \{x : |x - x^1| < \varepsilon \quad \text{and } u(x) = u(x^1)\}$$

is an open subset of some sphere $\{x : |x - z| = \rho_1\}$, $z \in \Omega$, $\rho_1 > 0$, and

$$\frac{\partial u}{\partial \rho}(x) = \frac{\partial u}{\partial \rho}(x^1) < 0 \quad \forall x \in \Gamma_1, \quad (\rho : \text{radial distance from } z).$$

Let $\hat{\Gamma}_1$ denote the maximal connected component of the set

$$\left\{x : \ |x - z| = \rho_1 \text{ and } \frac{\partial u}{\partial \rho}(x) < 0\right\},$$

containing the point x^1. Then, proceeding as before, we obtain that $\hat{\Gamma}_1$ is relatively open in $\{x : |x - z| = \rho_1\}$ and

$$\frac{\partial u}{\partial \rho}(x) = \frac{\partial u}{\partial \rho}(x^1) \qquad \forall x \in \hat{\Gamma}_1.$$

This means that $\hat{\Gamma}_1$ is relatively closed in $\{x : |x - z| = \rho_1\}$. Thus we have

$$\hat{\Gamma}_1 = \{x : |x - z| = \rho_1\}.$$

We can repeat these arguments for all points of $\overline{U_1}$. Since $u \in C^1(\{0 < u < \sup u\})$ we infer that u is radially symmetric in $B_R(z) \setminus \overline{B_r(z)}$ for some $R > r \geq 0$ and $(\partial u/\partial \rho)(x) < 0\ \forall x \in B_R(z) \setminus \overline{B_r(z)}$. Note that, if $\{u > 0\}$ is unbounded, then possibly $R = +\infty$. The Theorem is proved. ■

Next we give a purely *analytic* description of local symmetry in terms of continuous symmetrization.

Theorem 6.2. *Let Ω be an open set with $\Omega = \Omega^*$ and $u \in W_{0+}^{1,p}(\Omega)$ for some $p \in [1, +\infty)$. Further, let G be a strictly convex Young function satisfying* (3.43). *Finally, let u be continuously differentiable on $\{x : 0 < u(x) < \sup u\}$ and suppose that this last set is open. Then, if*

$$\lim_{t \searrow 0} \frac{1}{t}\left(\int_{\mathbb{R}^n} G(|\nabla u|)\mathrm{d}x - \int_{\mathbb{R}^n} G(|\nabla u^t|)\mathrm{d}x\right) = 0, \tag{6.12}$$

u is locally symmetric in direction y.

Proof. Let x^1, x^2 satisfy (6.1) and (6.2) and let U_1 be as in Remark 6.1. We have $u_y(x^2) \leq 0$. First assume that $u_y(x^2) < 0$. There are small neighbourhoods W_1, W_2 of the points x^1 and x^2, respectively, such that $u_y > 0$ in W_1 and $u_y < 0$ in W_2. Let $y_i = y_i(x', u)$, $i = 1, 2$, denote the corresponding inverse functions which exist for every (x', u) lying in a small neighbourhood V of the point $(x_0', u(x_0', y_1))$. Then the function u^t can be represented by corresponding inverse functions y_i^t, $i = 1, 2$, according to the formulas (2.10) for sufficiently small values $t > 0$, say $0 < t < t_0$. Let $G_i(t)$ denote the images of V in the (x', y)-domain after the mappings $(x', u) \longmapsto (x', y_i^t(x', u))$, $i = 1, 2$. Note that $G_1(0) \subset W_1$ and $G_2(0) \subset W_2$.

We approximate u by good functions which coincide with u in the domains $G_i(0)$, $i = 1, 2$. By proceeding as in the proof of Theorem 3.1 we infer that

$$\int_{\Omega\setminus(G_1(0)\cup G_2(0))} G(|\nabla u|)\mathrm{d}x \geq \int_{\Omega\setminus(G_1(t)\cup G_2(t))} G(|\nabla u^t|)\mathrm{d}x \tag{6.13}$$

and

$$\begin{aligned} I(t) &:= \int_{G_1(t)\cup G_2(t)} G(|\nabla u^t|)\mathrm{d}x \\ &= \sum_{k=1}^{2} \int_V G\left(\left\{1 + \sum_{i=1}^{n-1}\left(\frac{\partial y_k^t}{\partial x_i}\right)^2\right\}^{1/2} \left|\frac{\partial y_k^t}{\partial u}\right|^{-1}\right)\left|\frac{\partial y_k^t}{\partial u}\right|\mathrm{d}x'\mathrm{d}u \\ &\quad \forall t \in (0, t_0). \end{aligned} \tag{6.14}$$

We introduce the parameter $\lambda := (1/2)(1 - \mathrm{e}^{-t})$, $t \in [0, +\infty]$, and set $\psi(\lambda) := I(t)$. By setting $\psi(1-\lambda) := \psi(\lambda)\ \forall\lambda \in [0, (1/2)]$, we formally extend the definition of $\psi(\lambda)$ for all $\lambda \in [0, 1]$. Assume for a moment that $\psi(0) > \psi(1/2)$. Since $\psi(\lambda)$ is convex we obtain that $\lim_{t\searrow 0}(I(t) - I(0))/t < 0$. In view of (6.13) and (6.14) this contradicts to (6.12). Thus we have $\psi(0) = \psi(1/2)$. Since G is *strictly* convex we infer from this that $y_{1,x_i} = y_{2,x_i}$, $i = 1, \ldots, n-1$, and $y_{1,u} = -y_{2,u}$ almost everywhere in V. This means that (6.4) is satisfied throughout the domain $G_1(0)$.

Next let us assume that $u_y(x^2) = 0$. Since $u_y(x^1) > 0$, the implicit function theorem tells us that the problem

$$u(x_0', y) = u(x_0', y_1) + \varepsilon, \quad (x_0', y) \in G_1(0),$$

has a unique solution $y = y_1^\varepsilon$ if ε is positive and small enough, say $\varepsilon \in (0, \varepsilon_0)$. For $\varepsilon \in (0, \varepsilon_0)$ let y_2^ε denote the (unique!) number satisfying $y_1^\varepsilon < y_2^\varepsilon$ and $u(x_0', y_1^\varepsilon) = u(x_0', y_2^\varepsilon) < u(x_0', y)\ \forall y \in (y_1^\varepsilon, y_2^\varepsilon)$. Since u is differentiable we can choose a sequence $\varepsilon_m \searrow 0$ such that $u_y(x_0', y_2^{\varepsilon_m}) < 0$. Then from the earlier considerations follows that $u_y(x_0', y_1^{\varepsilon_m}) = -u_y(x_0', y_2^{\varepsilon_m})$. Clearly we have $\lim_{m\to\infty} y_i^{\varepsilon_m} = y_i$, $i = 1, 2$. Since $u \in C^1(\{0 < u < \sup u\})$ this yields

$$\lim_{m\to\infty} u_y(x_0', y_2^{\varepsilon_m}) = -u_y(x_0', y_1) < 0,$$

a contradiction. Thus the condition (6.4) is again satisfied for all $x = (x', y) \in G_1(0)$. Now set

$$\hat{G}_1 := \{(x', y) \in U_1 : u(x', y) = u(x', y_1 + y_2 - y) < u(x', z)\ \forall z \in (y, y_1 + y_2 - y)\}.$$

Obviously we have $G_1(0) \subseteq \hat{G}_1$, and we can argue as before to infer that $\hat{G}_1$ is relatively open in U_1. Let $\overline{x_m} = (\overline{x_m'}, \overline{y_m})$, $m = 1, 2, \ldots$, be any sequence in $\hat{G}_1$ converging to some point $\overline{x} = (\overline{x'}, \overline{y}) \in U_1$. Since $u \in C^1(\{0 < u < \sup u\})$ we have $u_y(\overline{x}) > 0$ and $u(\overline{x}) = u(\overline{x'}, y_1 + y_2 - \overline{y}) \leq u(\overline{x'}, y)\ \ \forall y \in (\overline{y}, y_1 + y_2 - \overline{y})$. Therefore we find some value $\hat{y} \in (\overline{y}, y_1 + y_2 - \overline{y}]$ such that $u(\overline{x}) = u(\overline{x'}, \hat{y}) < u(\overline{x'}, y)\ \ \forall y \in (\overline{y}, \hat{y})$. If $\hat{y} < y_1 + y_2 - \overline{y}$, then $u_y(\overline{x'}, \hat{y}) = 0$. This is impossible by the earlier considerations. Thus $\hat{y} = y_1 + y_2 - y$, i.e. $\overline{x} \in \hat{G}_1$. Therefore $\hat{G}_1$ is relatively closed with respect to U_1. But this means that $\hat{G}_1 = U_1$. The Theorem is proved. ■

Analogously one can prove the following extension of Theorem 6.2.

COROLLARY 6.1

Let u satisfy the assumptions of Theorem 6.2 *with the corresponding integrals replaced by the more general ones in* (3.18), *where the functions* $G, a, a_{ij},\ i, j = 1, \ldots, n-1$, *are as in Corollary* 3.3 *and* $G(x', v, z)$ *is strictly convex in z. Then the conclusions of Theorem* 6.2 *hold.*

7. Elliptic problems

Now we apply the preceding considerations to elliptic problems. First we deal with the variational problem (P) from the introduction.

Theorem 7.1. *Let* Ω *be a domain in* $\mathbb{R}^n$ *with* $\Omega = \Omega^*$. *For some* $p \in [1, +\infty)$ *let K be a closed subset of* $W^{1,p}_{0+}(\Omega)$ *and assume that K has the property that, if* $v \in K$, *then also* $v^t \in K$ *for every* $t \in [0, +\infty]$. *Let* $G = G(x', v, z)$ *be nonnegative and continuous on* $\mathbb{R}^{n-1} \times (\mathbb{R}^+_0)^2$, *and suppose that G is strictly convex in z and satisfies* (3.34). *Furthermore, let u be a local minimizer of problem* (P), *and suppose that* F, u *satisfy the assumptions of Theorem* 5.1. *Finally assume that the set* $\{0 < u < \sup u\}$ *is open and* $u \in C^1(\{0 < u < \sup u\})$.

Then u is locally symmetric in direction y.

Proof. From Theorem 5.1 we infer (5.4). In view of Corollary 3.3 and the inequality $J(u) \le J(u^t)$, $(t \in [0, +\infty])$, we obtain that

$$\int_\Omega G(x', u, |\nabla u|)\mathrm{d}x = \int_\Omega G(x', u^t, |\nabla u^t|)\mathrm{d}x \qquad \forall t \in [0, +\infty].$$

By Corollary 6.1 this means that u is locally symmetric in direction y. ■

By Corollary 3.3 and 6.1 the following extension of Theorem 7.1 is obvious.

COROLLARY 7.1

Let Ω, K, G, F, u *be as in Theorem* 7.1 *and, in addition, suppose that* $|\nabla v|$ *in* (1.1) *is replaced by the "generalized gradient" in* (3.18) *and the functions* a, a_{ij} *are as in Corollary* 3.3. *Then the conclusion of Theorem* 7.1 *holds.*

Theorem 7.1 yields the following Corollary 7.2 in the radially symmetric case.

COROLLARY 7.2

Let $\Omega = B_R$ *for some* $R > 0$ *or* $\Omega = \mathbb{R}^n$, *and let* K, G, F *and u be as in Theorem* 7.1. *In addition, suppose that G and the function B in* (5.2) *are independent of x, F satisfies* (5.3) *in every rotated coordinate system and*

$$F = F(|x|, v) \qquad \forall (x, v) \in \Omega \times [0, \sup u]. \tag{7.1}$$

Then u is locally symmetric in every direction.

Remark 7.1. (1) If F is independent of x, then we may relax the conditions on F in Theorem 7.1 and Corollaries 7.1 and 7.2. By Remark 5.2 it is enough to demand in this case that F is a Borel function and $F(u) \in L^1(\mathbb{R}^n)$. (2) If F is as in Corollary 7.2 and is differentiable in v, then $(\partial F)/(\partial v)$ is nondecreasing in $|x|$. (3) In view of (2.30), the assumption on K in Theorem 7.1 means that K may include side constraints of the types $(\lambda \in \mathbb{R}, c \geq 0, \mu > 0)$

$$\varphi \leq v \leq \psi, \qquad \text{where } \varphi = \varphi^*, \psi = \psi^*, \tag{7.2}$$

$$\int_\Omega g(v)\mathrm{d}x = \lambda, \qquad \text{where } g \text{ is a Borel function, or} \tag{7.3}$$

$$|\{v > c\}| = \mu. \tag{7.4}$$

In the case of the constraints (7.2) the statement of the problem allows to deal with 'ring-shaped' geometries. Note also that by the monotonicity of continuous symmetrization we infer from (7.2) that $\varphi = \varphi^t \leq v^t \leq \psi^t = \psi$, $(t \in [0, +\infty])$. Constraints of type (7.4) lead to variational solutions of *overdetermined* boundary value problems (see [Se] and [AC]). (4) Assume that $F(x, v)$ is Lipschitz continuous in v and

$$K = W_0^{1,p}(\Omega) \cap \{\text{constraints of the form (7.2)}\}.$$

Then K is *convex* and well-known analysis shows that a local minimizer u of (P) is a solution of the following (local) *variational inequality*

$$\int_\Omega G_z(x', u, |\nabla u|)|\nabla u|^{-1}\nabla u \nabla(v - u)\mathrm{d}x \geq \int_\Omega g(x)(v - u)\mathrm{d}x$$
$$\forall v \in K \text{ with } \|\nabla(v - u)\|_p < \varepsilon. \tag{7.5}$$

Here $g(\cdot) \in \partial_v F(\cdot, u(\cdot))$, $\partial_v F(x, v)$ is defined by (5.27) and ε is a given (small) constant. These well-known problems appear in models for reaction and diffusion processes (see [Di] and [K1]).

Remark 7.1(4) suggests to investigate directly the following *differential inclusion* instead of problem (P).

$$u \in W_{0+}^{1,p}(\Omega),$$
$$-\nabla(G_z(x, u, |\nabla u|)|\nabla u|^{-1}\nabla u) \in \tilde{f}(x, u). \tag{7.6}$$

Here $\tilde{f}(x, v)$ denotes the maximal monotone graph of $f(x, v)$ with respect to v (see (5.26)). The idea in proving symmetry results consists in using a Green-type identity with test function $(u^t - u)$ (namely (7.10) below) and to exploit the estimates of § 5 for small t. Clearly the assumptions on the data of the problem (7.6) and its solution will be more restrictive than in Theorem 7.1, especially in the case of unbounded domains.

Theorem 7.2. *Let Ω be a bounded domain in $\mathbb{R}^n$ with $\Omega = \Omega^*$, and let $G = G(x', z)$ be nonnegative, continuous in x', differentiable and strictly convex in z, and satisfies*

$$G(x', 0) = 0 \qquad \textit{and}$$
$$G_z(x', z) \leq C(1 + z^{p-1}) \qquad \forall (x', z) \in \mathbb{R}^{n-1} \times \mathbb{R}_0^+ \tag{7.7}$$

for some $p \in [1, +\infty)$ and $C > 0$. Furthermore, let $u \in W_{0+}^{1,p}(\Omega)$, let $f = f(x, v)$ measurable on $\Omega \times [0, \sup u]$, symmetrically nonincreasing in y and satisfies (5.11), *and suppose that f can be decomposed as follows*

$$
\begin{aligned}
& f = f_1 + f_2 + f_3, \qquad \textit{where} \\
& f_1 = f_1(x, v) \quad \textit{is continuous in } v, \\
& f_2 = f_2(x, v) \quad \textit{is nonincreasing in } v \textit{ and} \\
& f_3 = h(x)k(x', v), \quad \textit{with } h \textit{ and } k \textit{ as in Theorem } 5.2(\text{iv}).
\end{aligned} \tag{7.8}
$$

Finally let $u \in W_{0+}^{1,p}(\Omega)$ *satisfy weakly*

$$
-\nabla(G_z(x', |\nabla u|)|\nabla u|^{-1}\nabla u) = g \qquad \text{in } \Omega, \tag{7.9}
$$

where $g(\cdot) \in \tilde{f}(\cdot, u(\cdot))$, $\tilde{f}(x, v)$ *denotes the maximal monotone graph of* $f(x, v)$ *with respect to* v (*see* (5.26)). *In addition, suppose that the set* $\{0 < u < \sup u\}$ *is open and* $u \in C^1(\{0 < u < \sup u\})$. *Then* u *is locally symmetric in direction* y.

Proof. Let q be defined by $p^{-1} + q^{-1} = 1$. Since $u \in W_0^{1,p}(\Omega)$ we have $f(\cdot, u(\cdot)) \in L^q(\Omega)$ by our assumptions and thus also $g \in L^q(\Omega)$. From (7.10) we obtain the identity

$$
\int_\Omega G_z(x', |\nabla u|)|\nabla u|^{-1}\nabla u \nabla(u^t - u)\mathrm{d}x = \int_\Omega g(x)(u^t - u)\mathrm{d}x \quad \forall t \in [0, +\infty]. \tag{7.10}
$$

By using the convexity of G with respect to z and Corollary 3.3, we infer from this

$$
0 \geq \int_\Omega (G(x', |\nabla u^t|) - G(x', |\nabla u|))\mathrm{d}x \geq \int_\Omega g(x)(u^t - u)\mathrm{d}x \quad \forall t \in [0, +\infty]. \tag{7.11}
$$

We can estimate the right-hand side of (7.11) according to Theorem 5.2. This leads to

$$
\lim_{t \searrow 0} \frac{1}{t} \int_\Omega (G(x', |\nabla u^t|) - G(x', |\nabla u|))\mathrm{d}x = 0.
$$

Then the assertion follows by applying Corollary 6.1. ■

A similar result holds also for solutions of (7.9) in the entire space.

Theorem 7.3. *Let* $G = G(x', z)$ *be nonnegative, continuous in* x', *differentiable and strictly convex in* z *and let* G *satisfy*

$$
\begin{aligned}
& G(x', 0) = 0 \qquad \textit{and} \\
& G_z(x', z) \leq C z^{p-1} \qquad \forall (x', z) \in \mathbb{R}^{n-1} \times \mathbb{R}_0^+
\end{aligned} \tag{7.12}
$$

for some $p \in [1, +\infty)$ *and* $C > 0$. *Furthermore, let* f, u *be as in Theorem* 7.2 *with* $\Omega = \mathbb{R}^n$. *In addition, suppose that one of the following conditions* (i) *or* (ii) *is satisfied,*

(i) *u satisfies the decaying properties* (4.17)–(4.19) *of Theorem* 4.3;
(ii) *f satisfies* (5.22) *and u is positive and satisfies* (5.21).

Then u is locally symmetric in direction y.

Proof. By Green's formula we obtain for every $r > 0$

$$
\int_{B_r} G_z(x', |\nabla u|)|\nabla u|^{-1}\nabla u \nabla(u^t - u)\mathrm{d}x
$$

$$= \int_{B_r} g(x)(u^t - u)\mathrm{d}x + \int_{\partial B_r} G_z(x', |\nabla u|)|\nabla u|^{-1} \frac{\partial u}{\partial \nu}(u^t - u)\mathrm{d}S$$
$$\forall t \in [0, +\infty], \quad (\nu : \text{exterior normal}). \tag{7.13}$$

Furthermore, we have by Hölder's inequality,

$$I(r) := |\int_{\partial B_r} G_z(x', |\nabla u|)|\nabla u|^{-1} \frac{\partial u}{\partial \nu}(u^t - u)\mathrm{d}S| \le C\|\nabla u\|^{p-1}_{p,\partial B_r}\|u^t - u\|_{p,\partial B_r}, \tag{7.14}$$

for some number $C > 0$. Since $u \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ this means that $\lim_{r\searrow 0} I(r) = 0$. Hence, by passing to the limit $r \to +\infty$ in (7.13) we see that (7.11) holds with $\Omega = \mathbb{R}^n$. Then we argue as in the previous proof by applying Corollary 3.3, Theorem 5.2 and Corollary 6.1. in the case (i) and by using Lemma 5.1 in the case (ii). ■

Our method is also applicable to problems in ring-shaped domains.

COROLLARY 7.3

Let Ω, Ω_0 be two bounded domains in $\mathbb{R}^n$ with $\Omega = \Omega^*$, $\Omega_0 = \Omega_0^*$ and $\Omega_0 \subset\subset \Omega$. Furthermore, let G, f be as in Theorem 7.2 *and let $u \in W_0^{1,p}(\Omega)$ be a weak solution of the following problem*

$$-\nabla(G_z(x', |\nabla u|)|\nabla u|^{-1}\nabla u) = g, \quad 0 \le u \le 1 \quad \text{in } \Omega \setminus \Omega_0,$$
$$u \equiv 1 \quad \text{in } \overline{\Omega_0}, \tag{7.15}$$

where g is as in Theorem 7.2. *In addition, suppose that the set $\{0 < u < 1\}$ is open and that $u \in C^1(\{0 < u < 1\})$. Then u is locally symmetric in direction y.*

Proof. Since $\Omega_0 = \Omega_0^*$, we see that $(u^t - u) \in W_0^{1,p}(\Omega \setminus \overline{\Omega_0})$ for every $t \in [0, +\infty]$. Therefore the identity (7.11) again holds and we can proceed exactly as in the proof of Theorem 7.2. ■

The proof of the following corollary is analogous.

COROLLARY 7.4

Let G, f, u be as in Corollary 7.3 *with Ω replaced by $\mathbb{R}^n$. In addition, suppose that one of the conditions* (i) *or* (ii) *of Theorem* 7.3 *is satisfied. Then u is locally symmetric in direction y.*

By using the Corollaries 3.3 and 6.1 we may extend the previous results to more general differential operators (with obvious changes in the proof).

COROLLARY 7.5

Let the functions G, a, a_{ij}, $i, j = 1, \ldots, n-1$, be as in Corollary 3.4 *and independent of v. Furthermore, let f, u be as in Theorem* 7.2 *or* 7.3 *with the equation* (7.9) *replaced by*

$$-\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(G_z\left(x', \left\{\sum_{j,k=1}^{n} a_{jk}u_{x_j}u_{x_k}\right\}^{1/2}\right)\left\{\sum_{j,k=1}^{n} a_{jk}u_{x_j}u_{x_k}\right\}^{-1/2}\sum_{j=1}^{n} a_{ij}u_{x_j}\right) = g. \tag{7.16}$$

(*Here* $x_n := y,\ a_{nn} := a^2$ *and* $a_{in} = a_{ni} := 0,\ i = 1, \dots, n-1.$) *Then the conclusions of Theorem* 7.2 *or* 7.3, *respectively, hold.*

In the 'isotropic' cases the following consequences of the above results are immediate.

COROLLARY 7.6

Let u satisfy the assumptions of Theorem 7.2, 7.3 *or of Corollary* 7.3, 7.4. *Suppose that the function G and the functions b and k in* (5.10)*, respectively* (7.8)*, are independent of x. Further, let* $\Omega = B_R$ *or* $\Omega = \mathbb{R}^n$ *and* $\Omega_0 = B_r$ *for some numbers* $R > r > 0$*, and suppose that*

$$f = f(|x|, v), \qquad f \text{ is nonincreasing in } |x|. \tag{7.17}$$

Then u is locally symmetric in every direction.

Let us give a typical example with discontinuous nonlinearity f which is covered by Theorem 7.3 and Corollary 7.6.

Example 7.1. Let $u \in W^{1,p}(\mathbb{R}^n)$ for some $p \in (1, +\infty)$, and let $\varphi = \varphi(x')$ be a measurable function on $\mathbb{R}^{n-1}$ satisfying

$$\varphi(x') \geq \delta \qquad \forall x' \in \mathbb{R}^{n-1} \quad \text{for some } \delta > 0. \tag{7.18}$$

Further let $u \in W^{1,p}(\mathbb{R}^n)$ satisfy

$$-\Delta_p u \equiv -\nabla(|\nabla u|^{p-2}\nabla u) = g, \quad u > 0 \qquad \text{in } \mathbb{R}^n, \tag{7.19}$$

where

$$g(x)\begin{cases} = 1 & \text{if } u(x) > \varphi(x') \\ \in [0,1] & \text{if } u(x) = \varphi(x') \\ = 0 & \text{if } u(x) < \varphi(x') \end{cases} \qquad \forall x \in \mathbb{R}^n. \tag{7.20}$$

From (7.20) we see that $g(\cdot) \in \tilde{f}(\cdot, u(\cdot))$, where $\tilde{f}(x, v)$ is the maximal monotone graph of

$$f(x, v) = \chi(\{v > \varphi(x')\}, \quad (x, v) \in \mathbb{R}^n \times [0, \sup u].$$

Note that, if $p = 2,\ n = 3$ and $\varphi = \varphi(|x'|)$, then the problem (7.19) can be seen as a model for an equilibrium configuration of incompressible axially symmetric rotating fluids or rotating stars. The fluid rotates about the y-axis, the function $f(\cdot, u(\cdot))$ represents the mass density of the fluid and the function φ comes from the (prescribed) rotational law (see [Lio2, F], [B3]).

In view of (7.18) and since $u \in L^p(\mathbb{R}^n)$ we have $g \in L^1(\mathbb{R}^n)$. Since g is bounded, this yields $\lim_{|x|\to\infty} u(x) = 0$. By (7.18) we infer that g has bounded support. Now we see that u satisfies the assumptions (and in particular (ii)) of Theorem 7.3. In the particular case $\varphi \equiv \delta$, (i.e. $f(x, v) = \chi(\{v > \delta\})$, u satisfies the assumptions of Corollary 7.6.

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References

[AL] Almgren J A and Lieb E H, Symmetric decreasing rearrangement is sometimes continuous. *J. Am. Math. Soc.* **2** (1989) 683–773

[Alt] Alt H-W, *Lineare Funktionalanalysis.* 2nd ed. (Berlin: Springer-Verlag) (1992)

[AC] Alt H-W and Caffarelli L, Existence and regularity for a minimum problem with free boundary. *J. Reine Angew. Math.* **325** (1981) 105–144

[ALT] Alvino A, Lions P-L and Trombetti G, On optimization problems with prescribed rearrangements. *Nonlinear Anal. T.M.A.* **13** (1989) 185–220

[BaN] Badiale M and Nabana E, A note on radiality of solutions of p-Laplacian equation. *Appl. Anal.* **52** (1994) 35–43

[Ba] Baernstein II A, A unified approach to symmetrization, in: Partial differential equations of elliptic type (eds) A Alvino *et al*. (1995) *Symposia matematica* (Cambridge Univ. Press) vol. 35 pp. 47–91

[BM] Bandle C and Marcus M, Radial averaging transformations and generalized capacities. *Math. Z.* **145** (1975) 11–17

[Be] Beckner W, Sobolev inequalities, the Poisson semigroup and analysis on the sphere S^n. *Proc. Nat. Acad. Sci. U.S.A.* **89** (1992) 4816–4819

[BeN] Berestycki H and Nirenberg L, The method of moving planes and the sliding method. *Bol. Soc. Brasil. Mat. (N.S.)* **22** (1991) 1–37

[BLL] Brascamp H J, Lieb E H and Luttinger J M, A general rearrangement inequality for multiple integrals. *J. Funct. Anal.* **17** (1974) 227–237

[B1] Brock F, Axially symmetric flow with finite cavities. *Z. Anal. Anw.* **12** (1993) I: 97–112, II: 297–303

[B2] Brock F, Continuous Steiner-symmetrization. *Math. Nachr.* **172** (1995) 25–48

[B3] Brock F, Continuous polarization and Symmetry of Solutions of Variational Problems with Potentials. in: Calculus of variations, applications and computations, Pont-a-Mousson 1994 (eds) C Bandle *et al*, *Pitman Research Notes in Math.* **326** (1995) 25–35

[B4] Brock F, *Continuous rearrangement and symmetry of solutions of elliptic problems*, Habilitation Thesis (Leipzig) (1998) pp. 124

[B5] Brock F, Weighted Dirichlet-type inequalities for Steiner-symmetrization. *Calc. Var.* **8** (1999) 15–25

[B6] Brock F, Radial symmetry for nonnegative solutions of semilinear elliptic problems involving the p-Laplacian, in: Progress in partial differential equations, Pont-á-Mousson 1997, vol. 1 (eds) H Amann *et al*, *Pitman Research Notes in Math.* **383**, 46–58

[BS] Brock F and Solynin A Yu, An approach to symmetrization via polarization. preprint, (Köln) (1996) pp. 60, to appear in *Trans. A.M.S.*

[BZ] Brothers J and Ziemer W P, Minimal rearrangements of Sobolev functions. *J. Reine Angew. Math.* **384** (1988) 153–179

[Bu] Burchard A, Steiner symmetrization is continuous in $W^{1,p}$. *GAFA* **7** (1997) 823–860

[C] Coron J-M, The continuity of rearrangement in $W^{1,p}(\mathbb{R})$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **11(4)** (1984) 57–85

[Dam] Damascelli L, Comparison theorems for some quasilinear degenerate elliptic operators and applications to symmetry and monotonicity results. *Ann. Inst. H. Poincaré, Anal. non linéaire* **15** (1998) 493–516

[DamPa] Damascelli L and Pacella F, Monotonicity and symmetry of solutions of p-Laplace equations, $1 < p < 2$, via the moving plane method (1998) pp. 22 to appear in: *Ann. Scuola Norm. Sup. Pisa.*

[Da] Dancer E N, Some notes on the method of moving planes. *Bull. Austr. Math. Soc.* **46** (1992) 425–434

[Di] Diaz J I, Nonlinear PDE and free boundaries, vol. I, Elliptic equations. *Pitman Research Notes* (Boston) (1985) vol. 106

[Du] Dubinin V N, Capacities and geometric transformations in n-space. *GAFA* **3** (1993) 342–369

[EG] Evans L C and Gariepy R F, Measure theory and fine properties of functions. (London: CRC Press) (1992)

[F] Friedman A, *Variational principles and free-boundary problems* (NY: Wiley-Interscience) (1982)

[GNN] Gidas B, Ni W-M and Nirenberg L, Symmetry and related properties via the maximum principle. *Comm. Math. Phys.* **68** (1979) 209–243

[GT] Gilbarg D and Trudinger N S, Elliptic Partial Differential Equations of Second Order. 2nd ed., (Berlin: Springer-Verlag) (1983)

[GKPR] Grossi M, Kesavan S, Pacella F and Ramaswamy M, *Symmetry of positive solutions of some nonlinear equations.* preprint (Roma) (1995)

[K1] Kawohl B, Rearrangements and convexity of level sets in PDE. *Springer Lecture Notes* (1985) vol. 1150

[K2] Kawohl B, On the simple shape of stable equilibria, in: Geometry of Solutions to Partial Differential Equations (ed.) G Talenti (Academic Press) (1989) 73–89

[K3] Kawohl B, On the isoperimetric nature of a rearrangement inequality and its consequences for some variational problems. *Arch. Rat. Mech. Anal.* **94** (1986) 227–243

[K4] Kawohl B, On starshaped rearrangement and applications. *Trans. Amer. Math. Soc.* **296** (1986) 377–386

[KP] Kesavan S and Pacella F, Symmetry of positive solutions of a quasilinear elliptic equation via isoperimetric inequalities. *Appl. Anal.* **54** (1994) 7–37

[Lio1] Lions P-L, Two geometrical properties of solutions of semilinear problems. *Appl. Anal.* **12** (1981) 267–272

[Lio2] Lions P-L, Minimization problems in $L^1(\mathbb{R}^3)$. *J. Funct. Anal.* **41** (1981) 236–275

[M] Marcus M, Radial averaging of domains, estimates for Dirichlet integrals and applications. *J. d' Analyse* **27** (1974) 47–93

[McN] McNabb A, Partial Steiner symmetrization and some conduction problems. *J. Math. Anal. Appl.* **17** (1967) 221–227

[Sa] Sarvas J, Symmetrization of condensers in *n*-space. *Ann. Acad. Sci. Fenn. Ser. A1* **522** (1972) 1–44

[Se] Serrin J, A symmetry problem in potential theory. *Arch. Rat. Mech. Anal.* **43** (1971) 304–318

[SeZ] Serrin J and Zou H, Symmetry of ground states of quasilinear elliptic equations, preprint (1998) pp. 24

[So] Solynin A Yu, Continuous symmetrization of sets. *Zapiski Nauchnykh Seminarov LOMI Akademii Nauk SSSR* **185** (1990) 125–139