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Radial symmetry for nonnegative solutions of semilinear elliptic equations involving the p-Laplacian

Notation

$\mathbb{R}_0^+$	$\{x \in \mathbb{R} : \ x \geq 0\}$
$\lvert x\rvert$	$\sqrt{x_1^2 + \ldots x_n^2}$ if $\mathbb{R}^n \ni x = (x_1, \ldots, x_n)$
∇	gradient

1. Preliminaries

Let B be a ball in $\mathbb{R}^n$, centered at the origin, $p > 1$, and let

$$f \in C(\mathbb{R}_0^+ \times \mathbb{R}_0^+), \quad f = f(r, v) \quad \text{nonincreasing in } r\ , \ (r \geq 0, v \geq 0). \tag{1}$$

We consider weak solutions u of the following problem

$$(\mathbf{P}) \qquad \begin{aligned} & u \in W_0^{1,p}(B) \cap C^1(\overline{B}), \\ & u \geq 0, \ u \not\equiv 0, \ -\Delta_p u \equiv \nabla(|\nabla u|^{p-2}\nabla u) = f(|x|, u) \quad \text{in } \ B, \end{aligned} \tag{2}$$

and we ask for conditions on the nonlinearity f, under which u is *radially symmetric*. The following result is well-known from the celebrated paper of Gidas, Ni and Nirenberg, [GNN1]:

If $p = 2$, $u \in C^2(\overline{B})$, $u > 0$ and f admits a decomposition

$$\begin{aligned} f &= f_1 + f_2, \quad \textit{where} \\ f_1 &= f_1(r, v) \quad \textit{is Lipschitz-continuous in } v, \\ f_2 &= f_2(r, v) \quad \textit{is nondecreasing in } v, \ \ (r \geq 0, \ v \geq 0), \end{aligned} \tag{3}$$

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then u is radially symmetric and radially decreasing, i.e.

$$\begin{aligned} &u = u(r), \quad (r = |x|), \\ &\frac{\partial u(x)}{\partial r} < 0, \qquad \text{if} \;\; x \in B \setminus \{0\}. \end{aligned} \tag{4}$$

The proof of this result uses the so-called *moving plane method* which turned out to be a very powerful technique in showing that positive solutions of some boundary value problems in symmetric domains are symmetric (see e.g. [GNN1,2],[BeN1,2],[Da], [Li1,2],[LN] and the literature cited therein). The method exploits the invariance of the equation with respect to reflections and maximum and comparison principles for *uniformly* elliptic operators. It should be pointed out that the moving plane technique is not restricted to classical solutions, due to some refinements in [BeN2] and [Da]. For instance, one can show that the above result holds true, if u is merely in $C^1(\overline{B})$.
However, if $p \neq 2$ in problem **(P)** - or, more generally, in case of *degenerate* elliptic operators - the moving plane device is applicable only under additional assumptions on f and u (see [BaN],[Dam]). Therefore there where some attempts in the literature to prove the symmetry of the solutions by other means.
Combining an isoperimetric inequality and a Pohozaev-type identity Kesavan and Pacella [KP] showed that solutions of **(P)** satisfy (4) provided that $p = n$ and f is positive and independent of x. Note that this idea is due to P.-L. Lions who gave a proof of this result in the special case of two dimensions. However, this method is not applicable if $p \neq n$.
In this paper we obtain new symmetry results for solutions of **(P)** and also for solutions of a similar problem in the entire space. Our assumptions on the nonlinearity f are much weaker then in (3): In the Theorems 1 and 2 below the function f merely satisfies some *growth condition in neighbourhoods of its zero points.* Our approach is based on the Lemmata 1 and 1' below. These *weak* symmetry results were recently obtained by the author, by using a new rearrangement technique called *continuous Steiner symmetrization* (CStS) (see [Br1,2]). It would go beyond the scope of this article to give the definition of CStS here, since its construction is quite complicated. But since that material is relatively new, let us explain some ideas.

Given a Banach space X of measurable functions (e.g. $L^p(\mathbb{R}^n)$ or $W_0^{1,p}(\mathbb{R}^n)$, $p \in$

$[1, +\infty)$), and a unit vector $e \in \mathbb{R}^n$, the CStS in direction e - or CStS, in short - is a continuous homotopy $t \longmapsto v^t$, $0 \le t \le +\infty$, which connects $v \in X$ with its Steiner symmetrization with respect to the hyperplane $\{x : \ x \cdot e = 0\}$, v^*, such that $v^0 = v$ and $v^\infty = v^*$. Several integral inequalities of the form

$$\int_{\mathbb{R}^n} F(x, v, \nabla v)\, dx \ge \int_{\mathbb{R}^n} F(x, v^t, \nabla v^t)\, dx, \quad 0 < t \le +\infty, \tag{5}$$

were proved in [Br1,2], e.g. for $F(x, v, \nabla v) \equiv |\nabla v|^p$. Note also, that, if $F(x, v, \nabla v) \equiv G(v)$ with G continuous, then (5) holds with the equality sign.
Furthermore, there holds a useful symmetry criterion (see [Br2], Theorem 6.2):

Let $u \in W_0^{1,p}(B) \cap C^1(\overline{B})$, $u \ge 0$ and

$$\int_B \left(|\nabla u^t|^p - |\nabla u|^p \right) dx = o(t) \qquad \text{as } t \searrow 0. \tag{6}$$

Then u satisfies some weak - called *local* - kind of symmetry which can be described roughly as follows: Every connected component of the subset $\{(x, u(x)) : \ 0 < u(x) < \sup u, \ e \cdot \nabla u \ne 0\}$ of the graph of u finds a congruent counterpart after reflection about some hyperplane $\{x : \ x \cdot e = \lambda\}$, $\lambda \in \mathbb{R}$.

Note that the proof of this criterion is quite delicate and depends on the construction of CStS and on the convexity of the integrands in (6). The criterion can be used to "identify symmetric situations" for local minimizers in some variational problems. Furthermore, by exploiting the structure of the equation in (2), one can show that solutions of **(P)** satisfy (6) for CStS's in *arbitrary directions* e, ($|e| = 1$). Then some purely geometric observations lead to the following

Lemma 1 *(see [Br2], Theorem 7.2)*
Let u be a solution of **(P)**. *Then B permits the following decomposition,*

$$B = \bigcup_{k=1}^{m} C_k \cup \{x : \nabla u(x) = 0\}, \qquad \textit{where} \tag{7}$$

$$C_k = B_{R_k}(z_k) \setminus \overline{B_{r_k}(z_k)}, \quad (z_k \in B, \ 0 \le r_k < R_k), \tag{8}$$

$$u = u(\rho), \quad (\rho = |x - z_k|), \quad \textit{and}$$

$$\frac{\partial u(x)}{\partial \rho} < 0 \quad \forall x \in C_k, \tag{9}$$

$$u(x) \geq u\Big|_{\partial B_{r_k}(z_k)} \quad \forall x \in B_{r_k}(z_k), \tag{10}$$
$$(k = 1, \dots, m),$$

and $m \in \mathbb{N} \cup \{+\infty\}$.

Remark 1 In accordance with [Br2], we will call *any* function $u \in C^1(\overline{B})$ which vanishes on ∂B and satisfies the conditions (7)-(10), *locally symmetric*.
Note that, since $u \in C^1(\overline{B})$, (7) and (9) imply

$$\nabla u = 0 \quad \text{on} \quad B \cap \partial C_k, \quad (k = 1, \dots, m). \tag{11}$$

By using well-known means as the maximum principle and the principle of unique continuation one can infer further symmetries of the solutions from their local symmetry in various cases. Below we cite a result from [Br2] which will not be used here, since the symmetry proofs in this paper are based on some new observations and lead to results which are more general.

Lemma 2 *(see [Br2], Theorem D)*
Let u *be a solution of* **(P)**, *and suppose that one of the following conditions* **(i)** *or* **(ii)** *is satisfied, (* $v \geq 0,\ r \geq 0$ *),*

(i) $f(r, v) > 0$;

(ii) $u > 0$ *in* B *in case that* $n = 1$ *and*

$f = f_1 + f_2$, *where*

$f_2 = f_2(r, v)$ *is nondecreasing in* v *and either*

(**a**) $p \in (1, 2]$ *and* $f_1 = f_1(r, v)$ *satisfies a uniform Hölder condition in* v *with exponent* $p - 1$, *or*

(**b**) $p > 2$ *and* $f_1 \equiv 0$.

Then (4) holds.

We will essentially use a strong maximum principle for the p-Laplacian which is due to Vazquez [V]. We mention that we need only its "boundary point version" Lemma 3' below.

Definition 1 *A function β belongs to the class $\mathcal{A}_p$, ($\beta \in \mathcal{A}_p$), if $\beta \in C(I\!R_0^+)$, $\beta(0) = 0$, β is nondecreasing, and either*

$$\textbf{(i)} \qquad \beta(S) = 0 \quad \text{for some} \quad S > 0, \quad \text{or} \tag{12}$$

$$\textbf{(ii)} \qquad \int_0^1 (s\beta(s))^{-1/p}\, ds = +\infty. \tag{13}$$

Lemma 3 *Let Ω be a domain in $I\!R^n$, $u \in L^\infty(\Omega)$, $\Delta_p u \in L^1_{loc}(\Omega)$ in the sense of distributions in Ω,*

$$u \geq 0, \quad -\Delta_p u + \beta(u) \geq 0 \quad \text{a.e. in} \ \ \Omega \tag{14}$$

where $\beta \in \mathcal{A}_p$. Then either $u = 0$ a.e. in Ω or $u > 0$ in Ω in the sense that for every compact $K \subset \Omega$ there is a constant $c = c(K) > 0$, such that $u \geq c$ a.e. in K.

Lemma 3' *Let Ω, u, β as in Lemma 3 and let x_0 be a point on $\partial\Omega$ satisfying the interior sphere condition. Let B be one such sphere and ν the corresponding interior normal at x_0. Then there exists $\gamma > 0$, such that*

$$\text{ess }\liminf \frac{u(x)}{(x - x_0)\cdot \nu} \geq \gamma \quad \text{as} \quad x \to x_0,\ x \in B; \tag{15}$$

in particular, if $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$, we have

$$\frac{\partial u}{\partial \nu}(x_0) \geq \gamma. \tag{16}$$

Remark 2 Let $\beta \in \mathcal{A}_p$. Since for every $s > 0$ we have

$$(s/2)\beta(s/2) \leq \int_0^s \beta(t)\, dt \leq s\beta(s),$$

due to the monotonicity of β, (13) can be reformulated as

$$\textbf{(ii')} \qquad \int_0^1 \Big(\int_0^s \beta(t)\, dt\Big)^{-1/p} ds = +\infty. \tag{17}$$

The condition (13) means that β must not be very large near $s = 0$. (13) is satisfied if

$$\beta(s) \leq cs^{p-1}$$

for a certain $c > 0$ and $0 < s < 1$. Note that for $p = 2$ we recover then the classical strong maximum principles for the Laplacian in Lemma 3 and 3'.

But (13) is also satisfied by β's for which $\beta(s)/s^{p-1}$ is not bounded at 0, for instance if

$$\beta(s) \leq s^{p-1}|\log s|^p, \quad (0 < s < 1).$$

2. Radial symmetry in a ball

Theorem 1 *Let u be a solution of problem* **(P)** *and suppose that*

(a) *if $f(R,V)=0$ for some $V>0$ and $R \geq 0$, then there is a function $\beta \in \mathcal{A}_p$, such that*

$$f(r,v) \leq \beta(V-v) \quad \text{for } 0 \leq v \leq V \text{ and } r \geq 0. \tag{18}$$

Then

$$\text{supp } u = \overline{\bigcup_{k=1}^{m} B_{R_k}(z_k)}, \tag{19}$$

where $z_k \in \mathbb{R}^n$, $R_k > 0$, u is radially symmetric in $B_{R_k}(z_k)$ with respect to the origin z_k, i.e.

$$\begin{aligned} &u = u(\rho), \quad (\rho = |x - z_k|), \quad \text{and} \\ &\frac{\partial u(x)}{\partial \rho} < 0 \quad \forall x \in B_{R_k}(z_k) \setminus \{z_k\}, \qquad (k = 1, \ldots, m), \end{aligned} \tag{20}$$

and the case $m=\infty$ is possible in (19). Furthermore, if in addition either one of the following conditions **(b1)** *or* **(b2)** *is satisfied,*

(b1) *$u > 0$ in B ;*

(b2) *if $f(R,0)=0$ for some $R \geq 0$, then there is a function $\beta \in \mathcal{A}_p$, such that*

$$-f(r,v) \leq \beta(v) \quad \text{for } v \geq 0, \ r \geq 0, \quad \text{and} \tag{21}$$

if $n=1$ and $f(R,0)<0$ for some $R \geq 0$, then $u>0$ in B ;

then (4) holds.

The proof of Theorem 1 is based on the following observation.

Lemma 4 *Let u be as in Lemma 1. Let $C = B_R(z) \setminus \overline{B_r(z)}$ be an annulus of the* *decomposition (7) and $x_0 \in \partial C \cap B$. Then, if $x_0 \in \partial B_r(z)$ and $r > 0$, or if $x_0 \in \partial B_R(z)$ and $n \geq 2$, we have*

$$f(|x_0|, u(x_0)) = 0. \tag{22}$$

Proof : First suppose that $x_0 \in \partial B_r(z)$ and $r > 0$. Since, by (10), $u(x) \geq u(x_0)$ in $B_r(z)$ and $\nabla u(x_0) = 0$, we can apply Lemma 3' to infer that $f(|x_0|, u(x_0)) \leq 0$. On the other hand, since $u(x) \leq u(x_0)$ in C, Lemma 3' gives also $f(|x_0|, u(x_0)) \geq 0$.
Next suppose that $x_0 \in \partial B_R(z)$ and $n \geq 2$. Since $u(x) \geq u(x_0)$ in C and $\nabla u(x_0) = 0$, Lemma 3' yields $f(|x_0|, u(x_0)) \leq 0$. It remains to show

$$f(|x_0|, u(x_0)) \geq 0. \tag{23}$$

Assume that

$$\text{there is a subsequence } \{C_{k'}\} \text{ of annuli in (7), such that} \\ \lim_{k' \to \infty} z_{k'} = x_0 \quad \text{and} \quad \lim_{k' \to \infty} R_{k'} = 0. \tag{24}$$

Since $u(x) > u\big|_{\partial B_{R_{k'}}(z_{k'})}$ in $B_{R_{k'}}(z_{k'})$, the maximum principle tells us that there are points $y_{k'} \in B_{R_{k'}}(z_{k'})$ such that $0 \leq -\Delta_p u(y_{k'}) = f(|y_{k'}|, u(y_{k'}))$. Since $\lim_{k' \to \infty} y_{k'} = x_0$, we obtain (23) in this case.
Now suppose that (24) does not hold. Using the decomposition (7) we see that either

(i) there is an annulus $C_1 = B_{R_1}(z_1) \setminus \overline{B_{r_1}(z_1)}$ in (8) with $x_0 \in \partial B_{R_1}(z_1)$ and $\nabla u(x) = 0$ in $B_\varepsilon(x_0) \setminus (C \cup C_1)$ for some $\varepsilon > 0$, or

(ii) $\nabla u(x) = 0$ in $B_\varepsilon(x_0) \setminus C$ for some $\varepsilon > 0$.

Clearly in both cases **(i)** and **(ii)** we have $u = u(x_0)$ on some open subset of B, which means that $f(|x_0|, u(x_0)) = 0$. □

Proof of Theorem 1: By Lemma 1, u is locally symmetric. We claim that the annuli C_k in (7) are in fact *punctured balls*, i.e. we have in (8)

$$r_k = 0, \quad (k = 1, \ldots, m). \tag{25}$$

Let $C = B_R(z) \setminus \overline{B_r(z)}$, $(R > r)$, be one of the annuli in (8) and suppose that $r > 0$. By Lemma 4 we have $f(|x|, U) = 0 \quad \forall x \in \partial B_r(z)$, where $U = u\Big|_{\partial B_r(z)}$. We set $w := U - u$. Assumption **(a)** yields

$$\begin{aligned}
&w \in W^{1,p}(C) \cap C^1(\overline{C}),\\
&w \geq 0, \quad -\Delta_p w + \beta(w) \geq 0 \quad \text{in } \ C,\\
&w = |\nabla w| = 0 \quad \text{on } \ \partial B_r(z),
\end{aligned}$$

where $\beta \in \mathcal{A}_p$. But this is impossible by Lemma 3'. Hence we must have $r = 0$. This proves (25) and the assertions (19),(20) follow.

Next we claim that

$$u = 0 \quad \text{on } \ \partial B_{R_k}(z_k), \quad (k = 1, \ldots, m). \tag{26}$$

We fix some $l \in \{1, \ldots, m\}$, $l \neq \infty$, and set

$$u_l(x) := \begin{cases} u\Big|_{\partial B_{R_k}(z_k)} & \text{if } \ x \in B_{R_k}(z_k), \quad (k = 1, \ldots, m, \ k \neq l) \\ u(x) & \text{otherwise} \end{cases}.$$

Then $u_l \in C^1(\overline{B})$, since $u \in C^1(\overline{B})$. Hence $\nabla u_l = 0$ in $B \setminus B_{R_l}(z_l)$ which means that $u_l(x) = u\Big|_{\partial B_{R_l}(z_l)} \quad \forall x \in B \setminus B_{R_l}(z_l)$. Since $u_l \leq u$, this implies that $u = 0$ on $\partial B_{R_l}(z_l)$, and (26) follows. This proves (4) in the case **(b1)**.

Next suppose **(b2)** and assume that for some $k \in \{1, \ldots, m\}$,

$$B_{R_k}(z_k) \neq B. \tag{27}$$

Since $u \in C^1(\overline{B})$, this implies

$$|\nabla u| = 0 \quad \text{on } \partial B_{R_k}(z_k). \tag{28}$$

We split into two cases.

(i) Let $n \geq 2$. Then we have $f(|x|, 0) = 0$ on $\partial B_{R_k}(z_k)$ by Lemma 4. By assumption, there is a function $\beta \in \mathcal{A}_p$, such that $-\Delta_p u + \beta(u) \geq 0$ in $B_{R_k}(z_k)$. Together with (28), this contradicts to Lemma 3'.

(ii) Let $n = 1$. In view of (28), the strong maximum principle yields $f(|x|, 0) \le 0$ on $\partial B_{R_k}(z_k)$. Furthermore, there cannot be $f(|x_0|, 0) < 0$ at some point $x_0 \in \partial B_{R_k}(z_k)$ in view of **(b2)** and (26). Hence we must have $f(|x|, 0) = 0$ on $\partial B_{R_k}(z_k)$. Proceeding as in case **(i)**, we obtain again a contradiction to Lemma 3'. □

Remark 3 Recalling Remark 2, it is easy to check that Theorem 1 contains Lemma 2 as a special case.

3. Radial symmetry for entire solutions

Next we consider weak solutions of the following problem

$$(\mathbf{P}_\infty) \qquad \begin{aligned} & u \in W^{1,p}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n), \\ & u \ge 0, \ u \not\equiv 0, \ -\Delta_p u = f(|x|, u) \quad \text{in} \ \ \mathbb{R}^n, && (29) \\ & u(x) \longrightarrow 0 \qquad \text{as} \ \ |x| \to \infty. && (30) \end{aligned}$$

Since every solution u of $(\mathbf{P}_\infty)$ with *bounded support* is also a solution of **(P)** with $B = B_R(0)$ and $R > 0$ large enough, Theorem 1 yields the following

Corollary 1 *Let u be a solution of problem $(\mathbf{P}_\infty)$ with bounded support, and suppose that f satisfies condition* **(a)** *of Theorem 1. Then (19) and (20) hold.*

Remark 4 Solutions of $(\mathbf{P}_\infty)$ with bounded support play a certain role in describing reaction-diffusion processes where u stands for the stationary concentration of a chemical (see [Di],[KKL]). The set $\{u \equiv 0\}$ is then often called the "dead core". Note that dead cores can appear if f does not satisfy the conditions **(b1)**,**(b2)** of Theorem 1. This is the case, if for instance

$$f(t, 0) = 0 \quad \text{and} \quad f(t, v) \le -cv^q \quad \text{for} \ \ 0 < v < 1, \ t \ge 0,$$

for certain constants $c > 0$ and $q \in (0, p-1)$.

Next we investigate *positive* solutions of $(\mathbf{P}_\infty)$ satisfying certain decaying conditions at infinity.

Theorem 2 *Let u be a positive solution of problem $(\mathbf{P}_\infty)$, and let f satisfy condition*

(a) *of Theorem 1. In addition, suppose that either*

$$\begin{aligned} &\textbf{(i)} \quad f(t,v) \text{ is nonincreasing in } v \text{ for } 0 < v < \delta \text{ for some } \delta > 0 \\ &\qquad \text{and } f(|\cdot|, u(\cdot)) \in L^1(\mathbb{R}^n), \text{ or} \end{aligned} \tag{31}$$

$$\begin{aligned} &\textbf{(ii)} \quad u \text{ satisfies, } (\ \alpha \in (0,1),\ R,\ d > 0\ ), \\ &u(x) \ \geq \varphi(R/\alpha) \quad \text{if } \ |x| \leq R, \quad \text{and} \\ &\left.\begin{array}{rl} \varphi(|x|/\alpha) \leq \quad u(x) & \leq \varphi(|x|), \\ |\nabla u(x)| & \leq d|x|^{-1}\varphi(|x|) \end{array}\right\} \quad \text{if } \ |x| \geq R, \end{aligned} \tag{32}$$

where $\varphi : \mathbf{R}^+ \longrightarrow \mathbf{R}^+$ *is decreasing and continuous and satisfies*

$$\int_0^{+\infty} r^{(n/p)-1}\varphi(r)\; dr < +\infty.$$

Then there is some point $x_0 \in \mathbb{R}^n$ *such that*

$$\begin{aligned} &u = u(\rho), \quad (\rho = |x - x_0|), \quad \text{and} \\ &\frac{\partial u(x)}{\partial \rho} < 0 \quad \text{if } \ x \in \mathbb{R}^n \setminus \{x_0\}. \end{aligned} \tag{33}$$

Remark 5 It is easy to verify that u satisfies the assumptions **(ii)** of Theorem 2 if, ($R > 1,\ \delta, c_1, c_2, c_3, \gamma, \lambda > 0,\ \sigma, \tau \in \mathbf{R}$),

$$u(x) \geq \delta \qquad \text{for } \ |x| \leq R \tag{34}$$

and either

$$(\alpha) \qquad \gamma > (n/p) \qquad \text{and} \tag{35}$$

$$\left.\begin{array}{rl} c_1|x|^{-\gamma}\Big(\log|x|\Big)^{-\sigma} \leq \quad u(x) & \leq c_2|x|^{-\gamma}\Big(\log|x|\Big)^{-\sigma} \\ |\nabla u(x)| & \leq c_3|x|^{-\gamma-1}\Big(\log|x|\Big)^{-\sigma} \end{array}\right\} \ \text{if } \ |x| \geq R, \quad \text{or}$$

$$(\beta) \qquad \left.\begin{array}{rl} c_1 e^{-\lambda|x|}|x|^{\tau} \leq \ u(x) \leq & c_2 e^{-\lambda|x|}|x|^{\tau} \\ |\nabla u(x)| & \leq c_3 e^{-\lambda|x|}|x|^{\tau} \end{array}\right\} \ \text{if } \ |x| \geq R. \tag{36}$$

The proof of Theorem 2 is based on the following analogue of Lemma 1 see [Br2], Theorem 7.3).

Lemma 1' *Let* u *be a solution of problem* $(\mathbf{P}_\infty)$ *satisfying either one of the conditions*

(i) *or* **(ii)** *of Theorem 2. Then u satisfies (7)-(10) with B replaced by $\mathbb{R}^n$. Furthermore, the case $R_1 = +\infty$ is possible in (8).*

Proof of Theorem 2: By assumption, u is as in Lemma 1'. Proceeding as in the proof of Theorem 1, we infer that $r_k = 0$, $(k = 1, \ldots, m)$, which means that $u = 0$ on $\partial B_{R_k}(z_k)$, $(k = 2, \ldots, m)$. But since $u > 0$, this implies $m = 1$, and (33) follows. $\Box$

Remark 6 The Lemmata 1,1' and 3' remain valid for "p-Laplacian-like" operators

$$\begin{aligned} \mathcal{L}u \ \equiv\ & \nabla\Big(g(|\nabla u|)|\nabla u|^{-1}\nabla u\Big), \qquad \text{where} \\ & g \in C(\mathbb{R}_0^+),\ g \ \text{ is strictly increasing and} \\ & cz^{p-1} \le g(z) \le Cz^{p-1} \quad \text{for } z \ge 0 \text{ and } C > c > 0, \end{aligned} \tag{37}$$

(see [V] and [Br2]). Therefore Theorem 1,2 and Corollary 1 hold true for Δ_p replaced by operators $\mathcal{L}$ satisfying (37).

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