Nonlinear Differ. Equ. Appl. (2016) 23:36

1021-9722/16/030001-16
published online June 7, 2016
DOI 10.1007/s00030-016-0390-1

Nonlinear Differential Equations and Applications NoDEA

Symmetry for a general class of overdetermined elliptic problems

F. Brock

Abstract. Let Ω a bounded domain in $\mathbb{R}^N$, and let $u \in C^1(\overline{\Omega})$ a weak solution of the following overdetermined BVP: $-\nabla(g(|\nabla u|)|\nabla u|^{-1}\nabla u) = f(|x|, u)$, $u > 0$ in Ω and $u = 0$, $|\nabla u(x)| = \lambda(|x|)$ on $\partial\Omega$, where $g \in C([0, +\infty) \cap C^1((0, +\infty))$ with $g(0) = 0$, $g'(t) > 0$ for $t > 0$, $f \in C([0, +\infty) \times [0, +\infty))$, f is nonincreasing in $|x|$, $\lambda \in C([0, +\infty))$ and λ is positive and nondecreasing. We show that Ω is a ball and u satisfies some "local" kind of symmetry. The proof is based on the method of continuous Steiner symmetrization.

Mathematics Subject Classification. 28D10, 35B05, 35B50, 35J25, 35J60, 35J65.

Keywords. Degenerate elliptic equation, Overdetermined boundary value problem, Symmetry of the solution, Continuous rearrangement.

1. Introduction

In a celebrated paper [35] Serrin proved the following symmetry result for an overdetermined elliptic boundary value problem:

Theorem A. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with C^2-boundary, and let $u \in C^2(\overline{\Omega})$ satisfy

$$Lu \equiv a(u, |\nabla u|)\Delta u + \sum_{i,j=1}^{N} b(u, |\nabla u|)u_{x_i}u_{x_j}u_{x_ix_j} = c(u, |\nabla u|),$$

$$u > 0 \quad \text{in} \quad \Omega, \tag{1.1}$$

$$u = 0, \ |\nabla u| = constant \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where a, b and c are continuously differentiable in each variable and L is uniformly elliptic. Then Ω is a ball and u is radially symmetric about the center of the ball.

The proof of Theorem A in [35] uses the so-called moving plane method which goes back to Alexandrov [1]. The method was first applied to the study of partial differential equations by Serrin [35], and it became very popular thanks to Gidas, Ni and Nirenberg's paper [18]. The method combines symmetry arguments and boundary versions of the strong maximum principle, and it has often been applied to show the symmetry of solutions in overdetermined problems, e.g. in [2,12,13,25,29,32–34,37,39]. The moving plane device applies to very general—even fully nonlinear—elliptic equations. On the other hand, if the equation degenerates and/or contains terms which are less regular, then the method often fails.

Several other tools have been applied in such situations. Let us give a short overview. One approach is based on a comparison principle which is combined with some Rellich-type identity, see e.g. [14,16,17,27,31,38]. Another idea, used e.g. in [5,28], is to exploit some integral identity which is equivalent to the overdetermined problem. Although these two tools can be applied to degenerate operators—for instance to the p-Laplacian—they are useful only for very special equations. A third method is based on a comparison with suitable radial solutions of the equation, and it is applicable to situations when the solution of the boundary value problem is unique, see e.g. [19,20,22]. A fourth approach is based on the method of domain derivative which has been widely investigated in shape optimization (see [36]). This device again seems useful in problems where the solution of the boundary value problem is unique (see [11,21]). Note, this approach also highlights the relation between the second ('overdetermined') boundary condition and minimization of appropriate domain functionals, see [3,4]. The method of domain derivative has also been combined with another tool: the so-called continuous Steiner symmetrization (CStS) (see [9,10]). The idea of CStS is to find "local analogues" to some well-known rearrangement inequalities (see [6,7,24,30]). The author exploited this method to prove symmetry results for nonnegative solutions of boundary value problems in symmetric domains (see [7,8]).

The aim of this paper is to give a new approach to overdetermined problems which is based on the CStS, but does not use domain derivatives. Although our method is restricted to operators in divergence form, we allow nonsmooth terms in the equation, and the solution of the boundary value problem need not be unique.

Let us fix some notation. By $x = (x_1, \ldots, x_N)$ we denote a point in $\mathbb{R}^N$, and by $|x|$ its norm. Our main result is:

Theorem 1. *Let $f : [0, +\infty) \times [0, +\infty) \to \mathbb{R}$ be a bounded measurable function, such that the mapping $v \mapsto f(r, v)$, $(v \in [0, +\infty))$, is continuous, uniformly for all r, and the mapping $r \mapsto f(r, v)$ is nonincreasing, $(r, v \in [0, +\infty))$. Let $g \in C([0, +\infty)) \cap C^1((0, +\infty))$ with $g(0) = 0$, $g'(t) > 0$ for $t > 0$, and let $\lambda \in C([0, +\infty))$ be a positive and nondecreasing function. Further, let Ω be a bounded smooth domain, and let $u \in C^1(\Omega) \cap C(\overline{\Omega})$ be a weak solution of the following problem,*

$$-\nabla\left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) = f(|x|, u),\ u > 0 \quad in \quad \Omega; \tag{1.3}$$

$$u = 0 \quad on \quad \partial\Omega; \tag{1.4}$$

given any $\varepsilon > 0$, there is an open set U_ε containing $\partial\Omega$ such that

$$||\nabla u(x)| - \lambda(|x|)| < \varepsilon \quad \forall x \in U_\varepsilon \cap \Omega. \tag{1.5}$$

Then Ω and u satisfy the following symmetry properties:
Ω is ball,

$$\Omega = \bigcup_{k=1}^{m} C_k \bigcup S, \qquad where \tag{1.6}$$

$C_k := \{x \in \Omega :\ r_k < |x - y^k| < R_k\}, \quad R_k > r_k \geq 0, \quad y^k \in \Omega,$
there is a function $v_k \in C^1((r_k, R_k))$ such that $u(x) = v_k(|x - y^k|) \quad \forall x \in C_k$,
$v_k'(\rho) < 0 \quad \forall \rho \in (r_k, R_k),$
$u(x) \geq u(y) \quad$ *if* $0 \leq |x - y^k| \leq |y - y^k| = r_k, \quad 1 \leq k \leq m, \qquad$ *and*
$\nabla u = 0 \quad$ *in* $\ S.$

The sets on the right-hand side of (1.6) are disjoint and there can be a countable number of annuli C_k, i.e. $m = +\infty$.

Remark 1. (a) (1.3) means:

$$\int_\Omega g(|\nabla u|)\frac{\nabla u \cdot \nabla v}{|\nabla u|}\, dx = \int_\Omega f(|x|, u)v\, dx \qquad \forall v \in W_0^{1,1}(\Omega) \cap W^{1,\infty}(\Omega), \tag{1.7}$$

where the expression $g(|y|)\frac{y}{|y|}$, $(y \in \mathbb{R}^N)$, is interpreted as the zero vector, if $y = 0$.

Note that if $\partial\Omega$ is smooth and $u \in C^1(\overline{\Omega})$, then (1.5) means

$$\frac{\partial u}{\partial \nu}(x) = \lambda(|x|) \quad \text{on } \partial\Omega, \quad (\nu : \quad \text{exterior normal}). \tag{1.8}$$

(b) Theorem 1 falls out of the scope of the above mentioned results for the following reasons:
- We do not assume that f is smooth in the second variable.
- The solution of the boundary value problem (1.3), (1.4) might be not unique.
- The differential operator in (1.3) is not assumed to be uniformly elliptic.

We also emphasize that the solution u is not radially symmetric in general. For instance, there are example of nonsymmetric solutions of problem (1.3,1.4) in a ball in the p-Laplacian case, that is, if $g(z) = z^{p-1}$ for some $p > 1$, (see [7,9]).

Now we outline the content of the article. In Sect. 2, we give the definition of CStS and we present some results of [6–8] which will be of later use. In Sect. 3 we prove Theorem 1 and we give some extensions of the result in Theorem 2.

2. Continuous Steiner symmetrization

For points $x \in \mathbb{R}^N$, $(N \geq 2)$, we write $x = (x_1, x')$, where $x' = (x_2, \ldots, x_N)$, and for any set $M \subset \mathbb{R}^N$ let $\chi(M)$ the characteristic function of M. If $u : \mathbb{R}^N \to \mathbb{R}$, then let $\{u > a\}$ and $\{b \geq u > a\}$ denote the sets $\{x \in \mathbb{R}^N : u(x) > a\}$, and $\{x \in \mathbb{R}^N : b \geq u(x) > a\}$, respectively, $(a, b \in \mathbb{R}, a < b)$. Let $\mathscr{L}^k$ denote k-dimensional Lebesgue measure, $(1 \leq k \leq N)$, and $\| \cdot \|_p$ the usual norm in $L^p(\mathbb{R}^N)$, $(1 \leq p \leq +\infty)$. By $\mathscr{M}(\mathbb{R}^N)$ we denote the family of Lebesgue measurable—measurable in short—sets in $\mathbb{R}^N$ with finite measure. Finally, let $\mathscr{S}_+(\mathbb{R}^N)$ denote the class of real, nonnegative measurable functions u satisfying

$$\mathscr{L}^N (\{u > c\}) < +\infty \quad \forall c > 0.$$

Note that nonnegative functions in $L^p(\mathbb{R}^N)$, $(1 \leq p < +\infty)$, belong to $\mathscr{S}_+(\mathbb{R}^N)$. Generally we treat measurable sets and functions in a.e. sense.

Given a unit vector $e \in \mathbb{R}^n$, a continuous Steiner symmetrization (CStS) is a continuous homotopy which connects sets $M \in \mathscr{M}(\mathbb{R}^N)$ and functions $u \in \mathscr{S}_+(\mathbb{R}^N)$ with their Steiner symmetrizations in direction e, M^*, respectively u^*. Homotopies of such type can be constructed in different ways (see [6,7] and the references cited therein). Below we define a variant of CStS which has been investigated by the author in [6,7].

For the convenience of the reader we first recall the definition of the well-known Steiner symmetrization (see e.g. [23]).

Definition 1. (*Steiner symmetrization*)

(i) For any set $M \in \mathscr{M}(\mathbb{R})$ let

$$M^* := \left(-\frac{1}{2}\mathscr{L}^1(M), \frac{1}{2}\mathscr{L}^1(M) \right)$$

denote the (one-dimensional) symmetrization of M.

(ii) Let $M \in \mathscr{M}(\mathbb{R}^N)$, $(N \geq 2)$. For every $x' \in \mathbb{R}^{N-1}$ let

$$M(x') := \{x_1 \in \mathbb{R} : (x_1, x') \in M\}.$$

The set

$$M^* := \left\{ x = (x_1, x') : x_1 \in (M(x'))^*, \ x' \in \mathbb{R}^{N-1} \right\} \tag{2.1}$$

is called the Steiner symmetrization of M (with respect to x_1).

(Note that M^* is symmetric and convex with respect to the hyperplane $\{x_1 = 0\}$.)

(iii) If $u \in \mathscr{S}_+(\mathbb{R}^N)$, $(N \geq 2)$, then the function

$$u^*(x) := \begin{cases} \sup \{c > 0 : x \in \{u > c\}^*\} & \text{if } x \in \bigcup_{c>0} \{u > c\}^* \\ 0 & \text{if } x \notin \bigcup_{c>0} \{u > c\}^* \end{cases}, \quad x \in \mathbb{R}^N, \tag{2.2}$$

is called the Steiner symmetrization of u (with respect to x_1).

(Note that $u^*(x_1, x')$ is symmetric with respect to $\{x_1 = 0\}$ and nonincreasing in x_1 for $x_1 > 0$.)

Definition 2. (*Continuous symmetrization of sets in* $\mathscr{M}(\mathbb{R})$)
A family of set transformations

$$E_t: \quad \mathscr{M}(\mathbb{R}) \longrightarrow \mathscr{M}(\mathbb{R}), \quad 0 \le t \le +\infty,$$

satisfying the properties, $(M, N \in \mathscr{M}(\mathbb{R}),\ 0 \le s, t \le +\infty)$

(i) $\mathscr{L}^1(E_t(M)) = \mathscr{L}^1(M)$, (equimeasurability),
(ii) If $M \subset N$, then $E_t(M) \subset E_t(N)$, (monotonicity),
(iii) $E_t(E_s(M)) = E_{s+t}(M)$, (semigroup property),
(iv) If M is an interval $[x-R, x+R]$, $(x \in \mathbb{R},\ R > 0)$, then $E_t(M) := [xe^{-t} - R, xe^{-t} + R]$,

is called continuous symmetrization.

The existence and uniqueness of the family E_t, $0 \le t \le +\infty$, has been proved in [7], Theorem 2.1.

Definition 3. [*Continuous Steiner symmetrization* (*CStS*)]

(i) Let $M \in \mathscr{M}(\mathbb{R}^N)$, $(N \ge 2)$. The family of sets

$$E_t(M) := \left\{x = (x_1, x') : \ x_1 \in E_t\left(M(x')\right),\ x' \in \mathbb{R}^{N-1}\right\}, \quad 0 \le t \le +\infty, \tag{2.3}$$

is called the continuous Steiner symmetrization (CStS) of M (with respect to x_1).

(ii) Let $u \in \mathscr{S}_+(\mathbb{R}^N)$. The family of functions $E_t(u)$, $0 \le t \le +\infty$, defined by

$$E_t(u)(x) := \begin{cases} \sup\{c > 0 : \ x \in E_t(\{u > c\})\} & \text{if } \ x \in \bigcup\limits_{c>0} E_t(\{u > c\}) \\ 0 & \text{if } \ x \notin \bigcup\limits_{c>0} E_t(\{u > c\}) \end{cases}, \quad x \in \mathbb{R}^N, \tag{2.4}$$

is called CStS of u with respect to x_1 in the case $N \ge 2$ and continuous symmetrization in the case $N = 1$.

Remark 2. 1. For convenience, we will henceforth simply write M^t and u^t for the sets $E_t(M)$, respectively for the functions $E_t(u)$, $(t \in [0, +\infty])$.

2. It can be shown that, if $M \in \mathscr{M}(\mathbb{R}^N)$ and M is open, then the sets M^t, $(t \in [0, +\infty])$, have open representatives. This makes it possible to give pointwise definitions of open sets and of continuous functions:

(i) If $M \in \mathcal{M}(\mathbb{R})$ is open and $t \in [0, +\infty]$, then let

$$M^{t,O} := \bigcup \left\{U : \ U \text{ is an open representative of } N^t,\ N \text{ open},\ N \subset\subset M\right\}. \tag{2.5}$$

(ii) If $M \in \mathcal{M}(\mathbb{R}^N)$, $(N \ge 2)$, is open and $t \in [0, +\infty]$, then let

$$M^{t,O} := \left\{x = (x_1, x') : \ x_1 \in (M(x'))^{t,O},\ x' \in \mathbb{R}^{N-1}\right\}. \tag{2.6}$$

Note that the relations (2.5), (2.6) have to be understood in pointwise sense. The sets $M^{t,O}$ in (2.5) and (2.6) are open and they are called the precise representatives of M^t.

(iii) If $u \in \mathscr{S}_+(\mathbb{R}^N)$ is continuous and $t \in [0, +\infty]$, then there exists a unique continuous representative of u^t which is given by (2.4)—now in pointwise sense!—where the sets $\{u > c\}^t$ have to be replaced by their precise representatives.

From now on let us agree that, if we speak about the CStS of open sets or continuous functions then we always mean their precise representatives.

Remark 3. Below we summarize basic properties of CStS, which have been proved by the author in [6,7], $(M \in \mathscr{M}(\mathbb{R}^N), u, v \in \mathscr{S}_+(\mathbb{R}^N),\ t \in [0, +\infty])$.

1. *Equimeasurability* From Definitions 2 and 3 we have

$$\mathscr{L}^N(M) = \mathscr{L}^N(M^t) \quad \text{and} \quad \{u^t > c\} = \{u > c\}^t \quad \forall c > 0. \tag{2.7}$$

2. *Monotonicity* If $u \leq v$ then $u^t \leq v^t$, [see [6], Theorem 5, formula (48)].
3. If $\psi : [0, +\infty) \to [0, +\infty)$ is bounded and nondecreasing with $\psi(0) = 0$, then

$$\psi(u^t) = \left(\psi(u)\right)^t, \tag{2.8}$$

[see [6], Theorem 8, formula (61)].

4. *Homotopy*

We have

$$M^0 = M, \quad u^0 = u, \quad M^\infty = M^*, \quad u^\infty = u^*. \tag{2.9}$$

(see [6], Theorems 1 and 4). Furthermore, from the construction of the CStS it follows that, if $M = M^*$ or $u = u^*$, then $M = M^t$, respectively $u = u^t$ for all $t \in [0, +\infty]$. Finally, if $t_n \to t$ as $n \to +\infty$ and $u \in L^p(\mathbb{R}^N)$ for some $p \in [1, +\infty)$, then

$$\lim_{n \to \infty} \|u^{t_n} - u^t\|_p = 0, \tag{2.10}$$

(see [7], Lemma 4.1).

5. *Cavalieri's principle* If F is continuous and if $F(u) \in L^1(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} F(u)\, dx = \int_{\mathbb{R}^N} F(u^t)\, dx, \tag{2.11}$$

[see [6], Theorem 8, formula (62)].

6. *Nonexpansivity in L^p* If $u, v \in L^p(\mathbb{R}^N)$ for some $p \in [1, +\infty)$ then

$$\|u^t - v^t\|_p \leq \|u - v\|_p, \tag{2.12}$$

(see [6], Lemma 3).

7. *Hardy-Littlewood inequality* If $u, v \in L^2(\mathbb{R}^N)$ then

$$\int_{\mathbb{R}^N} u^t v^t\, dx \geq \int_{\mathbb{R}^N} uv\, dx, \tag{2.13}$$

(see [6], Lemma 4).

8. Let $u \in L^\infty(\mathbb{R}^N)$ and suppose that u vanishes outside some ball B_R, $R > 0$. Furthermore, suppose that $F = F(x, v)$ is bounded and measurable on $B_R \times [0, +\infty)$, continuously differentiable in v with $F(x, 0) = 0 \ \ \forall x \in \mathbb{R}^N$, and $(\partial/\partial v)F(x, v)$ is even in x_1 and nonincreasing in x_1 for $x_1 > 0$. Then
$$\int_{B_R} F(x, u)\, dx \leq \int_{B_R} F(x, u^t)\, dx, \tag{2.14}$$
(see [7], Theorem 5.1).
9. If u is Lipschitz continuous with Lipschitz constant L, then u^t is Lipschitz continuous, too, with Lipschitz constant less than or equal to L (see [6], Theorem 11).
10. If supp $u \subset B_R$ for some $R > 0$, then we also have supp $u^t \subset B_R$. If, in addition, u is Lipschitz continuous on $\mathbb{R}^N$ with Lipschitz constant L, then we have
$$|u^t(x) - u(x)| \leq LRt \quad \forall x \in B_R, \tag{2.15}$$
(see [7], Theorem 4.2). Furthermore, there holds
$$\int_{B_R} G(|\nabla u^t|)\, dx \leq \int_{B_R} G(|\nabla u|)\, dx, \tag{2.16}$$
for every convex function $G : [0, +\infty) \to [0, +\infty)$ with $G(0) = 0$, (see [7], Corollary 3.3).

Note that **1.–3.**, **5.–7.**, **9.** and (2.16) are common properties of many rearrangements (see [23]).

The following symmetry criterion, proved in [7], Theorem 6.2, under the hypothesis $\Omega = \Omega^*$, holds even without that assumption.

Lemma 1. *Let Ω be a bounded open set, $u \in C^1(\overline{\Omega})$, $u > 0$ in Ω, $u = 0$ on $\partial\Omega$. Furthermore, let $G : [0, +\infty) \to [0, +\infty)$ be strictly convex with $G(0) = 0$, and suppose that*
$$\lim_{t \to 0} \frac{1}{t} \left(\int_\Omega G(|\nabla u|)\, dx - \int_{\Omega^t} G(|\nabla u^t|)\, dx \right) = 0. \tag{2.17}$$
Then u satisfies the following symmetry property:

If $y = (y_1, y') \in \mathbb{R}^N$ with
$$0 < u(y) < \sup u, \quad \frac{\partial u}{\partial x_1}(y) > 0, \tag{2.18}$$
and $\widetilde{y}$ is the (unique) point satisfying
$$\widetilde{y} = (\widetilde{y_1}, y'), \quad \widetilde{y_1} > y_1, \quad u(y) = u(\widetilde{y}) < u(z, y') \quad \forall z \in (y_1, \widetilde{y_1}), \tag{2.19}$$
then
$$\frac{\partial u}{\partial x_i}(y) = \frac{\partial u}{\partial x_i}(\widetilde{y}), \quad i = 1, \dots, N-1, \quad \text{and}$$
$$\frac{\partial u}{\partial x_1}(y) = -\frac{\partial u}{\partial x_1}(\widetilde{y}). \tag{2.20}$$

For the geometrical meaning of local symmetry, see [7], *Remark* 6.1.

We will say that u is locally symmetric in direction x_1 if u satisfies the properties (2.18)–(2.20).

Lemma 2. (see [7], Theorem 6.1) *Let Ω, u and G be as in Lemma 1, and suppose that for arbitrary rotations $x \longmapsto y = (y_1, y')$ of the coordinate system, u is locally symmetric in direction y_1. Then Ω is an at most countable union of disjoint open balls and u satisfies the symmetry properties (1.6).*

3. Symmetry of the solution

In this section we show Theorem 1. The idea is to use appropriate test functions v in (1.7) and then to use Lemma 1. This works well in the case of Steiner symmetric domains Ω (see [7]), choosing $v = u^t$, respectively $v = u$, (u^t: CStS of u, with small $t > 0$). However, in our situation, Ω is not assumed to be symmetric, so that u^t might not vanish on $\partial\Omega$. Therefore we modify the approach of [7], using appropriate cut-off functions of u^t and u.

First we introduce some notation. For functions v we write $v_+ := \max\{v, 0\}$. By the symbol $o(t)$ we denote any function satisfying $\lim_{t\to 0} o(t)/t = 0$ and which may vary from line to line. For any point $x \in \overline{\Omega}$ we write

$$d(x) := \text{dist}\,\{x; \partial\Omega\} \equiv \inf\{|x - z| : \, z \in \partial\Omega\}.$$

Throughout this section, let u be a solution of problem (1.3)–(1.5). For convenience, we extend u by zero outside Ω, so that $u \in C^{0,1}(\mathbb{R}^N)$. We denote by L the Lipschitz constant of u, and we set $u_0 := \max\{u(x) : \, x \in \Omega\}$. We choose a number $R > 0$ such that $\overline{\Omega} \subset B_R$, and we set $f_0 := \sup\{|f(|x|, v)| : \, |x| \leq R, \, 0 \leq v \leq u_0\}$ and

$$k := 2RL.$$

Finally, we fix some coordinate system

$$x = (x_1, x'), \quad (x_1 \in \mathbb{R}, \, x' \in \mathbb{R}^{N-1}).$$

Let u^t, $(0 \leq t \leq +\infty)$, denote the CStS of u with respect to x_1. Since $u \in C^{0,1}(\mathbb{R}^N)$, we have by Remark 3, **9.** and by (2.15),

$$u^t \in C^{0,1}(\mathbb{R}^N), \tag{3.1}$$

$$u^t \text{ has Lipschitz constant less than or equal to } L, \text{ and} \tag{3.2}$$

$$|u^t(x) - u(x)| \leq LRt \qquad \forall t \in [0, +\infty], \tag{3.3}$$

Next we obtain some estimates for u and u^t near the boundary of Ω.
Since u is positive in Ω and continuous on $\mathbb{R}^N$, and since $\overline{\Omega} \subset B_R$, we have

$$\lim_{s\to 0} \sup\,\{d(x) : 0 < u(x) \leq s\} = 0. \tag{3.4}$$

Lemma 3. *Let $t \in [0, +\infty)$ and $u^t(x) > kt$. Then $x \in \Omega$.*

Proof. We have by property (3.3),

$$u(x) \geq = -|u(x) - u^t(x)| + u^t(x) \geq -RLt + kt = RLt > 0.$$

Hence $x \in \Omega$. $\square$

For convenience, we set

$$M_1(t) := \{0 < u \le kt\} \quad \text{and} \tag{3.5}$$

$$M_2(t) := \Omega \cap \{0 < u^t \le kt\}, \quad (t \in (0, +\infty)). \tag{3.6}$$

Note that $M_1(t), M_2(t) \subset \Omega$, and by Remark 3, **1.**,

$$\mathscr{L}^N(M_1(t)) = \mathscr{L}^N(M_2(t)) \quad \forall t \in (0, +\infty). \tag{3.7}$$

Lemma 4. *There holds:*

$$\lim_{t\to 0} \sup \{d(x) : x \in M_1(t) \cup M_2(t)\} = 0. \tag{3.8}$$

Proof. In view of property (3.4) it is sufficient to show that

$$\lim_{t\to 0} \sup \{d(x) : x \in M_2(t)\} = 0. \tag{3.9}$$

Assume that (3.9) is not true. Then there exists a number $\delta > 0$, a sequence of points $\{x_n\} \subset \Omega$ and a decreasing sequence $\{t_n\} \subset \mathbb{R}$ with $\lim_{n\to\infty} t_n = 0$, such that $0 < u^{t_n}(x_n) \le kt_n$, but $d(x_n) \ge \delta$. The latter also implies that $u(x_n) \ge \varepsilon$, for some $\varepsilon > 0$. On the other hand, we have by (2.15),

$$\begin{aligned} u(x_n) &\le |u(x_n) - u^{t_n}(x_n)| + u^{t_n}(x_n) \\ &\le LRt_n + kt_n \to 0, \end{aligned}$$

as $n \to \infty$, a contradiction. $\square$

Lemma 5. *There exists a constant* $c_0 > 0$ *such that*

$$\mathscr{L}^N(M_1(t)) \le c_0 t, \quad (0 < t < +\infty). \tag{3.10}$$

Proof. By (1.5) and (3.4) there exist positive numbers τ and t_0, such that

$$|\nabla u(x)| \ge \tau \quad \text{if } x \in M_1(t_0). \tag{3.11}$$

By the Implicit Function Theorem, we have that for every $s \in (0, kt_0]$, $\{u > s\}$ is an open subset of Ω with $\partial\{u > s\} = \{u = s\}$, and $\{u = s\}$ is locally a C^1-hypersurface. Integrating (1.3) over $\{u > s\}$, $(s \in (0, kt_0])$, Green's Theorem yields

$$\int_{\{u=s\}} g(|\nabla u|)\, d\mathscr{H}_{N-1}(x) = \int_{\{u>s\}} f(|x|, u)\, dx.$$

By (3.11) this implies

$$\int_{\{u=s\}} d\mathscr{H}_{N-1}(dx) \le \frac{f_0}{g(\tau)} \mathscr{L}^N(\Omega), \quad (s \in (0, kt_0]). \tag{3.12}$$

Using this and the co-area formula (see [15]), we obtain:

$$\begin{aligned} \mathscr{L}(M_1(t)) &= \int_0^{kt} \left(\int_{\{u=s\}} \frac{d\mathscr{H}_{N-1}(x)}{|\nabla u|} \right) ds \le \frac{1}{\tau} \int_0^{kt} \left(\int_{\{u=s\}} d\mathscr{H}_{N-1}(x) \right) ds \\ &\le \frac{f_0}{\tau g(\tau)} \mathscr{L}^N(\Omega) kt, \quad (t \in (0, t_0]), \end{aligned}$$

and the assertion follows. $\square$

Proof of Theorem 1 The functions $(u^t-kt)_+$ have compact support in $\overline{\Omega}$ for all $t\in[0,+\infty]$, by Lemma 3. Hence we derive from (1.3) the integral identities

$$\begin{aligned} 0 &= \int_\Omega g(|\nabla u|)\frac{\nabla u}{|\nabla u|}\cdot\nabla\left((u^t-kt)_+-(u-kt)_+\right)\,dx \\ &\quad -\int_\Omega f(|x|,u)\left((u^t-kt)_+-(u-kt)_+\right)\,dx \\ &=: I_1(t)-I_2(t), \qquad t\in[0,+\infty). \end{aligned} \tag{3.13}$$

First we claim that

$$I_2(t)\geq o(t). \tag{3.14}$$

To show (3.14), we split $I_2(t)$ for $0<t\leq u_0/k$:

$$\begin{aligned} I_2(t) &= \int_{M_1(t)}\left(f(|x|,u)-f(|x|,kt)\right)\left((u^t-kt)_+-(u-kt)_+\right)\,dx \\ &\quad +\int_\Omega f(|x|,(u-kt)_++kt)\left((u^t-kt)_+-(u-kt)_+\right)\,dx \\ &=: I_{21}(t)+I_{22}(t). \end{aligned} \tag{3.15}$$

By (3.3) we have

$$|(u^t(x)-kt)_+-(u(x)-kt)_+|\leq LRt \qquad \forall x\in\Omega. \tag{3.16}$$

It follows from Lemma 5 and (3.16), that for $0<t\leq u_0/k$,

$$|I_{21}(t)|\leq\frac{2}{\tau}f_0LRc_0t^2. \tag{3.17}$$

Further, in view of (2.8) we have

$$(u^t-kt)_++kt=\left((u-kt)_++kt\right)^t.$$

Hence we obtain by using Remark 3, **8.** and Taylor's Theorem,

$$\begin{aligned} 0 &\leq \int_\Omega\left(F(|x|,(u^t-kt)_++kt)-F(|x|,(u-kt)_++kt)\right)\,dx \\ &= \int_\Omega\int_0^1 f(|x|,u_\theta^t)\,d\theta\,\left((u^t-kt)_+-(u-kt)_+\right)\,dx, \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} F(r,v) &:= \int_0^v f(r,w)\,dw, \quad (r,v\geq 0), \quad \text{and} \\ u_\theta^t &:= (1-\theta)(u-kt)_++\theta(u^t-kt)_++kt, \quad (\theta\in[0,1]). \end{aligned}$$

Using (3.16), we obtain,

$$|u_\theta^t(x)-(u(x)-kt)_+-kt|\leq\theta|u^t(x)-u(x)|\leq\theta LRt \quad \forall x\in\mathbb{R}^N, \quad \theta\in[0,1].$$

Since the mapping $v \mapsto f(|x|, v)$ is continuous, uniformly in x, this implies

$$\lim_{t\to 0}\sup\left\{|f(|x|, u^t_\theta(x)) - f(|x|, (u(x)-kt)_+ + kt)| : x\in\Omega, \quad \theta\in[0,1]\right\} = 0. \tag{3.19}$$

Finally, (3.16) and (3.18) yield, for $0 < t \le u_0/k$,

$$\begin{aligned} I_{22}(t) &\ge \int_\Omega \left\{ f(|x|, (u-kt)_+ + kt) - \int_0^1 f(|x|, u^t_\theta)\, d\theta \right\} \left((u^t - kt)_+ - (u-kt)_+\right)\, dx \\ &\ge -\sup\left\{\left|f(|x|, u^t_\theta) - f(|x|, (u-kt)_+ + kt)\right| : x\in\Omega, \quad \theta\in[0,1]\right\}\cdot LRt\cdot \mathscr{L}^N(\Omega). \end{aligned} \tag{3.20}$$

Now (3.14) follows from (3.17), (3.19) and (3.20).
Next we estimate $I_1(t)$. Let

$$G(z) := \int_0^z g(s)\, ds \quad \text{and} \quad h(z) := G(z) - zg(z), \quad (z\ge 0).$$

Note that h is nonincreasing. Since G is convex, we have for $t\in(0, u_0/k]$,

$$\begin{aligned} I_1(t) &= \int_{\Omega\setminus M_2(t)} g(|\nabla u|)\frac{\nabla u\cdot\nabla u^t}{|\nabla u|}\, dx - \int_{\Omega\setminus M_1(t)} g(|\nabla u|)|\nabla u|\, dx \\ &\le \int_{\Omega\setminus M_2(t)} g(|\nabla u|)|\nabla u^t|\, dx - \int_{\Omega\setminus M_1(t)} g(|\nabla u|)|\nabla u|\, dx \\ &\le \int_{\Omega\setminus M_2(t)} \left(G(|\nabla u^t|) - G(|\nabla u|) + g(|\nabla u|)|\nabla u|\right)\, dx \\ &\quad - \int_{\Omega\setminus M_1(t)} g(|\nabla u|)|\nabla u|\, dx =: \tilde{I}_1(t). \end{aligned} \tag{3.21}$$

Since

$$\begin{aligned} \tilde{I}_1(t) &= \int_{\Omega\setminus M_2(t)} G(|\nabla u^t|)\, dx - \int_{\Omega\setminus M_1(t)} G(|\nabla u|)\, dx \\ &\quad + \int_{\Omega\setminus M_2(t)} h(|\nabla u|)\, dx - \int_{\Omega\setminus M_2(t)} h(|\nabla u|)\, dx, \end{aligned}$$

we further deduce from (3.21),

$$\begin{aligned} I_1(t) \le \tilde{I}_1(t) &= \int_{\Omega\setminus M_2(t)} G(|\nabla u^t|)\, dx - \int_{\Omega\setminus M_1(t)} G(|\nabla u|)\, dx \\ &\quad + \int_{M_2(t)} h(|\nabla u|)\, dx - \int_{M_1(t)} h(|\nabla u|)\, dx \\ &=: I_{11}(t) - I_{12}(t) + I_{13}(t) - I_{14}(t), \quad (t\in(0, u_0/k)). \end{aligned} \tag{3.22}$$

By Lemma 4 and (1.5) we have

$$\lim_{t\to 0}\sup\{||\nabla u(x)| - \lambda(|x|)| : x\in M_1(t)\cup M_2(t)\} = 0. \tag{3.23}$$

Furthermore, since λ is nondecreasing and since $h(z) := G(z) - zg(z)$, $(z\in[0,+\infty))$, is nonincreasing, the function

$$p(x) := h\left(\lambda(|x|)\right), \quad (x\in\mathbb{R}^N),$$

satisfies $p = p^* = p^t \quad \forall t \in [0, +\infty]$. Applying the Hardy-Littlewood inequality (2.13) we obtain for $t \in (0, u_0/k]$,

$$\begin{aligned}\int_{\Omega\setminus M_1(t)} p\, dx &= \int_{\mathbb{R}^N} p\chi(\Omega\setminus M_1(t))\, dx \\ &\le \int_{\mathbb{R}^N} p\chi\left(\Omega\setminus M_2(t)\right)\, dx = \int_{\Omega\setminus M_2(t)} p\, dx. \end{aligned} \tag{3.24}$$

In view of (3.24), (3.7), Lemma 5 and (3.23), we obtain for $t \in (0, u_0/k]$,

$$\begin{aligned} I_{13}(t) - I_{14}(t) = &-\int_{M_1(t)} p\, dx + \int_{M_2(t)} p\, dx + \int_{M_1(t)} \left(p(x) - h\left(|\nabla u|\right)\right)\, dx \\ &-\int_{M_2(t)} \left(p(x) - h\left(|\nabla u|\right)\right)\, dx \\ &\le 2\mathscr{L}^N\left(M_1(t)\right) \sup\left\{|p(x) - h\left(|\nabla u(x)|\right)| : \, x \in M_1(t) \cup M_2(t)\right\} \\ &\le 2c_0 t \sup\left\{|p(x) - h\left(|\nabla u(x)|\right)| : \, x \in M_1(t) \cup M_2(t)\right\} = o(t). \end{aligned} \tag{3.25}$$

In conclusion, we have by (3.13), (3.14), (3.22) and (3.25) for $t \in [0, u_0/k]$,

$$\int_\Omega \Big(G(|\nabla(u^t - kt)_+|) - G(|\nabla(u - kt)_+|)\Big)\, dx \ge o(t). \tag{3.26}$$

Now fix $\varepsilon \in (0, u_0]$. Setting $w(x) := \min\{(u(x) - kt)_+; \varepsilon - kt\}$ for $t \in (0, u_0/k]$ and $x \in \mathbb{R}^N$, we have by (2.16),

$$\int_\Omega G(|\nabla w^t|)\, dx \le \int_\Omega G(|\nabla w|)\, dx. \tag{3.27}$$

Since $w^t(x) = \min\{(u^t(x) - kt)_+; \varepsilon - kt\}$, $(x \in \mathbb{R}^N)$, (3.26) together with (3.27) gives for $t \in (0, u_0/k]$,

$$\begin{aligned} &\int_\Omega \Big(G(|\nabla(u^t - \varepsilon)_+|) - G(|\nabla(u - \varepsilon)_+|)\Big)\, dx \\ &\quad = \int_\Omega \left(G(|\nabla w|) - G(|\nabla w^t|)\right)\, dx + \int_\Omega \Big(G(|\nabla(u^t - kt)_+|) - G(|\nabla(u - kt)_+|)\Big)\, dx \\ &\quad \ge \int_\Omega \Big(G(|\nabla(u^t - kt)_+|) - G(|\nabla(u - kt)_+|)\Big)\, dx \ge o(t). \end{aligned} \tag{3.28}$$

In view of Lemma 1, $(u - \varepsilon)_+$ is locally symmetric in direction x_1. Since the same estimate (3.28) can be obtained for CStS in arbitrary directions, Lemma 2 tells us that $\{u > \varepsilon\}$ is an at most countable union of mutually disjoint balls and $(u - \varepsilon)_+$ satisfies the symmetry property (1.6) of Theorem 1. Since ε was arbitrary, u is locally symmetric in every direction, too. Moreover, since $\Omega = \bigcup_{\varepsilon > 0}\{u > \varepsilon\}$, and Ω is connected, it must be a ball. The Theorem is proved. □

Remark 4. It is often possible to derive the radial symmetry of the solution of (1.3), (1.4) from their local symmetry by using other well-known tools. For instance, in the case of the p-Laplacian, i.e. if $g(z) = z^{p-1}$ for some $p > 1$, u is radially symmetric if $f(|x|, \cdot)$ satisfies some growth conditions in neighbourhoods of its zero points (see [8], Theorem 1).

We mention three typical situations for a general g:

Theorem 2. *Let Ω, f, g, λ and u be as in Theorem 1, and suppose that one of the following conditions* **(a)**–**(c)** *is satisfied:*

(a) *f is nonnegative,*
(b) *The mapping $r \mapsto f(r,v)$ is strictly decreasing,*
(c) *f is independent of x and the mapping $w \mapsto f(w)$ is nonincreasing.*

Then u is radially symmetric and radially decreasing, i.e. $\Omega = B_R(y)$), for some $R > 0$ and $y \in \mathbb{R}^N$ and there is a function $v \in C^1([0,R))$ such that

$$u(x) = v(|x-y|), \quad v'(\rho) \leq 0 \quad \forall \rho \in [0,R). \tag{3.29}$$

Moreover, we have $y = 0$ in case **(b)**.

Proof. We use the notations of (1.6).

(a) If $k \in \{1,\ldots,m\}$, we have by Green's Theorem,

$$\int_{B_{r_k}(y^k)} f(|x|,u)\,dx = \int_{\partial B_{r_k}(y^k)} g(|\nabla u|)\,d\mathscr{H}_{N-1}(x) = 0,$$

which means that $f \equiv 0$ and hence $u = \text{const}$ in $B_{r_k}(y^k)$, $(k = 1,\ldots,m)$. Since u is positive we must then have $m = 1$. Moreover, if $r_1 > 0$, then we must have $u = \text{const}$ in $B_{r_1}(y^1)$.

(b) If the mapping $r \mapsto f(r,v)$ is strictly decreasing, then (1.3) shows that we must have $y^k = 0$, $(k = 1,\ldots,m)$, in (1.6), which proves (3.29), with $y = 0$.

(c) Let Σ be an arbitrary $(N-1)$-hyperplane containing the center of the ball Ω. Denote by H one of the two open halfspaces into which $\mathbb{R}^N$ is split by Σ, and let σ denote reflection in Σ. Define

$$v(x) := u(\sigma x), \quad (x \in \overline{\Omega \cap H}).$$

Since

$$\left(g(|y|)\frac{y}{|y|} - g(|z|)\frac{z}{|z|}\right) \cdot (y-z) \geq 0$$

for all vectors $y, z \in \mathbb{R}^N$, and since the mapping $w \mapsto f(w)$ is nonincreasing, we obtain

$$\begin{aligned} 0 &\leq \int_{\Omega \cap H_+} \left(g(|\nabla u|)\frac{\nabla u}{|\nabla u|} - g(|\nabla v|)\frac{\nabla v}{|\nabla v|}\right) \cdot \nabla(u-v)\,dx \\ &= \int_{\Omega \cap H_+} (f(u) - f(v))(u-v)\,dx \leq 0. \end{aligned}$$

It follows that $u = v$ in $\Omega \cap H$. Since Σ was arbitrary, the assertion follows. $\square$

Acknowledgments

I thank L. Damascelli, A. Henrot, W. Reichel and A. Wagner for helpful discussions.

References

[1] Alexandrov, A.D.: A characteristic property of the spheres. Ann. Mat. Pura Appl. **58**(1), 303–315 (1962)

[2] Aftalion, A., Busca, J.: Radial symmetry of overdetermined boundary-value problems in exterior domains. Arch. Rat. Mech. Anal. **143**, 195–206 (1998)

[3] Bandle, C., Wagner, A.: Domain derivatives for energy functionals with boundary integrals. Inequalities and applications 2010. Internat. Ser. Numer. Math., vol. 161, pp. 3–17. Springer, Birkhäuser, Basel (2012)

[4] Bandle, C., Wagner, A.: Second domain variation for problems with Robin Boundary Conditions. J. Optim. Theory Appl. **167**(2), 430–463 (2015)

[5] Bennett, A.: Symmetry in an overdetermined fourth order elliptic boundary value problem. SIAM J. Math. Anal. **17**, 1354–1358 (1986)

[6] Brock, F.: Continuous Steiner-symmetrization. Math. Nachr. **172**, 25–48 (1995)

[7] Brock, F.: Continuous rearrangement and symmetry of solutions of elliptic problems. Proc. Indian Acad. Sci. Math. Sci. **110**(2), 157–204 (2000)

[8] Brock, F.: Radial symmetry for nonnegative solutions of semilinear elliptic problems involving the p-Laplacian. In: Amann, H., et al. (eds.) Progress in partial differential equations. Pont-á-Mousson, vol. 383, pp. 46–58. Pitman Research Notes, UK (1997)

[9] Brock, F., Henrot, A.: A symmetry result for an overdetermined elliptic problem using continuous rearrangement and domain derivative. Rend. Circ. Mat. Palermo **51**(3), 375–390 (2002)

[10] Chatelain, T., Choulli, M., Henrot, A.: Some new ideas for a Schiffer's conjecture. In: Malanowski, K., et al. (eds.) Modelling and optimization of distributed parameter systems. Applications to engineering. Papers from the IFIP WG 7.2 conference, Warsaw 1995, pp. 90–97. Chapman and Hall, UK (1996)

[11] Choulli, M., Henrot, A.: Use of the domain derivative to prove symmetry results in p.d.e. Math. Nachr. **192**, 91–103 (1998)

[12] Colesanti, A.: A symmetry result for p-Laplacian equation via the moving planes method. Appl. Anal. **55**, 207–213 (1994)

[13] Dalmasso, R.: Symmetry problems for elliptic systems. Hokkaido Math. J. **25**, 107–117 (1996)

[14] Farina, A., Kawohl, B.: Remarks on an overdetermined boundary value problem. Calc. Var. Partial Differ. Equations **31**(3), 351–357 (2008)

[15] Federer, H.: Geometric measure theory. Grundlehren der Mathematischen Wissenschaften, vol. 153, Springer, New York (1969)

[16] Fragala, I., Gazzola, F., Kawohl, B.: Overdetermined problems with possibly degenerate ellipticity, a geometric approach. Math. Zeitschr. **254**(1), 117–132 (2006)

[17] Garofalo, N., Lewis, J.L.: A symmetry result related to some overdetermined boundary value problems. Am. J. Math. **111**, 9–33 (1989)

[18] Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry and related properties via the maximum principle. Commun. Math. Phys. **68**, 209–243 (1979)

[19] Greco, A.: Symmetry around the origin for some overdetermined problems. Adv. Math. Sci. Appl. **13**(1), 387–399 (2003)

[20] Greco, A.: Constrained radial symmetry for monotone elliptic quasilinear operators. J. Anal. Math. **121**, 223–234 (2013)

[21] Henrot, A., Philippin, G.A.: On a class of overdetermined eigenvalue problems. Math. Methods Appl. Sci. **20**(11), 905–914 (1997)

[22] Henrot, A., Philippin, G.A., Prebet, H.: Overdetermined problems on ring shaped domains. Adv. Math. Sci. Appl. **9**, 737–747 (1999)

[23] Kawohl, B.: Rearrangements and convexity of level sets in PDE, vol. 1150. Springer Lecture Notes, New York (1985)

[24] Kawohl, B.: On the simple shape of stable equilibria. In: Talenti, G., (ed.) Geometry of solutions to PDE, pp. 73–89. Academic Press, Cambridge (1989)

[25] Kumaresan, S., Prajapat, J.: Serrin's result for hyperbolic space and sphere. Duke Math. J. **91**, 17–28 (1998)

[26] Lewis, J.L., Vogel, A.: On some almost everywhere symmetry theorems. In: Lloyd, N.G., et al. (eds.) Nonlinear diffusion equations and their equilibrium states, Proc. 3rd Conf., 1989. Prog. Nonlinear Differ. Equ. Appl., vol. 7, pp. 347–374. Gregynog/UK (1992)

[27] Molzon, R.: Symmetry and overdetermined boundary value problems. Forum Math. **3**, 143–156 (1991)

[28] Payne, L.E., Schaefer, P.W.: Duality theorems in some overdetermined boundary value problems. Math. Methods Appl. Sci. **11**, 805–819 (1989)

[29] Philippin, G.A., Ragoub, L.: One some second order and fourth order elliptic overdetermined problems. ZAMP **46**, 188–197 (1995)

[30] Polya, G., Szegö, G.: Isoperimetric inequalities in mathematical physics. Ann. Math. Studies, vol. 27. Princeton University Press, Princeton (1951)

[31] Ragoub, L.: On an elliptic over-determined problem in dimension two. Aust. J. Math. Anal. Appl. **7**(2), 8 (2010) (Art. 25)

[32] Reichel, W.: Radial symmetry by moving planes for semilinear elliptic boundary value problems on annuli and other nonconvex domains. In: Bandle, C., et al. (eds.) Progress in partial differential equations, elliptic and parabolic problems. Pitman Research Notes, vol. 325, pp. 164–182 (1995)

[33] Reichel, W.: Radial symmetry for an electrostatic, a capillarity and some fully nonlinear overdetermined problems on exterior domains. uZeitschr. fr̈ Anal. Anw. **15**, 619–635 (1996)

[34] Reichel, W.: Radial symmetry for elliptic boundary value problems on exterior domains. Arch. Rat. Mech. Anal. **137**, 381–394 (1997)

[35] Serrin, J.: A symmetry theorem in potential theory. Arch. Ration. Mech. Anal. **43**, 304–318 (1971)

[36] Sokolowski, J., Zolesio, J.P.: Introduction to shape optimization: shape sensitity analysis, Springer Series in Computational Mathematics, vol. 10. Springer, Berlin (1992)

[37] Vogel, A.L.: Symmetry and regularity for general regions having a solution to certain overdetermined boundary value problems. Atti Sem. Mat. Fis Univ. Modena. **40**, 443–484 (1992)

[38] Weinberger, H.F.: Remark on the preceding paper of Serrin. Arch. Ration. Mech. Anal. **43**, 319–320 (1971)

[39] Willms, N.B., Gladwell, G., Siegel, D.: Symmetry theorems for some overdetermined boundary value problems in ring domains. ZAMP **45**, 556–579 (1994)

F. Brock
Department of Mathematics
University of Rostock
Ulmenstr. 69
18057 Rostock
Germany
e-mail: friedemann.brock@uni-rostock.de

Received: 22 January 2016.
Accepted: 17 May 2016.