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Weighted isoperimetric inequalities in cones and applications

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ABSTRACT

This paper deals with weighted isoperimetric inequalities relative to cones of $\mathbb{R}^N$. We study the structure of measures that admit as isoperimetric sets the intersection of a cone with balls centered at the vertex of the cone. For instance, in case that the cone is the half-space $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N : x_N > 0\}$ and the measure is factorized, we prove that this phenomenon occurs if and only if the measure has the form $d\mu = a x_N^k \exp\left(c\,|x|^2\right) dx$, for some $a > 0$, $k, c \geq 0$. Our results are then used to obtain isoperimetric estimates for Neumann eigenvalues of a weighted Laplace–Beltrami operator on the sphere, sharp Hardy-type inequalities for functions defined in a quarter space and, finally, via symmetrization arguments, a comparison result for a class of degenerate PDE's.

1. Introduction

This paper deals with weighted relative isoperimetric inequalities in cones of $\mathbb{R}^N$. Let ω be an open subset of $\mathbb{S}^{N-1}$, the unit sphere of $\mathbb{R}^N$, and Ω the cone

$$\Omega = \left\{ x \in \mathbb{R}^N : \frac{x}{|x|} \in \omega, x \neq 0 \right\}. \tag{1.1}$$

We consider measures of the type $d\nu = \phi(x)dx$ on Ω, where ϕ is a positive Borel measurable function defined in Ω. For any measurable set $M \subset \Omega$, we define the ν-measure of M

$$\nu(M) = \int_M d\nu = \int_M \phi(x)dx \tag{1.2}$$

and the ν-perimeter of M relative to Ω

$$P_\nu(M,\Omega) = \sup\left\{ \int_M \operatorname{div}\left(\mathbf{v}(x)\phi(x)\right) dx : \mathbf{v} \in C_0^1(\Omega,\mathbb{R}^N),\ |\mathbf{v}| \leq 1 \right\}.$$

We also write $P_\nu(M,\mathbb{R}^N) = P_\nu(M)$. Note that if M is a smooth set, then

$$P_\nu(M,\Omega) = \int_{\partial M\cap\Omega} \phi(x)\, d\mathcal{H}_{N-1}(x).$$

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The isoperimetric problem reads as

$$I_\nu(m) = \inf\{P_\nu(M, \Omega) : M \subset \Omega, \nu(M) = m\}, \quad m > 0. \tag{1.3}$$

One says that M is an isoperimetric set if $\nu(M) = m$ and $I_\nu(m) = P_\nu(M, \Omega)$.

We give necessary conditions on the function ϕ for having $B_R \cap \Omega$ as an isoperimetric set, in Section 2. Here and throughout the paper, B_R and $B_R(x)$ denote the ball of radius R centered at zero and at x, respectively. In Theorem 2.1 we prove that if $B_R \cap \Omega$ is an isoperimetric set for every $R > 0$, then

$$\phi = A(r)B(\Theta),$$

where $r = |x|$ and $\Theta = \frac{x}{|x|}$.

As an application of Theorem 2.1, we prove a sharp Hardy-type inequality for functions defined in $Q = \{x_1 > 0, x_N > 0\}$ involving a power-type weight, (see Theorem 2.2).

We are able to give an explicit expression of the density ϕ in some special cases. For instance, when Ω is the half space

$$\Omega = \mathbb{R}^N_+ = \left\{x = (x_1, \dots, x_N) \in \mathbb{R}^N : x_N > 0\right\}, \tag{1.4}$$

if ϕ is a smooth function with a factorized structure,

$$\phi(x) = \prod_{i=1}^{N} \phi_i(x), \tag{1.5}$$

and if $B_R \cap \mathbb{R}^N_+$ is an isoperimetric set, then

$$\phi(x) = a x_N^k \exp(c\,|x|^2), \tag{1.6}$$

for some numbers $a > 0$, $k \geq 0$ and $c \geq 0$, (see Theorem 2.3).

Section 3 is dedicated to the case $\Omega = \mathbb{R}^N$, and to the proof of the following Theorem, which is the main result of our paper.

Theorem 1.1. *Let μ be the measure defined by*

$$d\mu = x_N^k \exp(c\,|x|^2)dx, \quad x \in \mathbb{R}^N_+, \tag{1.7}$$

with $k, c \geq 0$, and let M be a measurable subset of $\mathbb{R}^N_+$ with finite μ-measure. Then

$$P_\mu(M) \geq P_\mu(M^\star),$$

where $M^\star = B_{r^\star} \cap \mathbb{R}^N_+$, with $r^\star$ such that $\mu(M) = \mu(M^\star)$.

The proof of Theorem 1.1 requires some technical effort, due to the degeneracy of the measure on the hyperplane $\{x_N = 0\}$.

Note that Theorem 1.1 is embedded in a wide bibliography related to the isoperimetric problems for "manifolds with density" (see, for instance, [1–9]). Further references will be given in Section 2.

It was shown in [10] that the isoperimetric set for measures of the type $y^k dxdy$, with $k \geq 0$ and $(x, y) \in \mathbb{R}^2_+$, is $B_R \cap \mathbb{R}^2_+$. In [11] Borell proved that balls centered at the origin are isoperimetric sets for measures of the type $\exp(c\,|x|^2)dx$ in $\mathbb{R}^N$ with $c \geq 0$ (see also [2,8] for this and related results).

In Section 4 we consider degenerate elliptic problems of the type

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = x_N^k \exp(c\,|x|^2) f(x) & \text{in } D \\ u = 0 & \text{on } \Gamma_+, \end{cases} \tag{1.8}$$

where D is a bounded open set in $\mathbb{R}^N_+$, whose boundary is decomposed into a part Γ_0, lying on the hyperplane $\{x_N = 0\}$ and a part Γ_+ contained in $\mathbb{R}^N_+$. (For precise definitions, see Section 4). Assume that $c, k \geq 0$, $A(x) = (a_{ij}(x))_{ij}$ is an $N \times N$ symmetric matrix with measurable coefficients satisfying

$$x_N^k \exp(c\,|x|^2)\,|\zeta|^2 \leq a_{ij}(x)\zeta_i\zeta_j \leq \Lambda x_N^k \exp(c\,|x|^2)\,|\zeta|^2, \quad \Lambda \geq 1, \tag{1.9}$$

for almost every $x \in D$ and for all $\zeta \in \mathbb{R}^N$. Assume also that f belongs to the weighted Hölder space $L^2(D, d\mu)$, where $d\mu$ is the measure defined in (1.7).

The type of degeneracy in (1.9) occurs, for $k \in \mathbb{N}$, when one looks for solutions to linear PDE's which are symmetric with respect to a group of $(k+1)$ variables (see, e.g., [12,10,13] and the references therein). The case of a non-integer k has been the object of investigation, for instance, in the generalized axially symmetric potential theory (see, e.g., [14] and the subsequent works of A. Weinstein).

We obtain optimal bounds for the solution to problem (1.8) using a symmetrization technique due to Talenti (see [15] and also [16–18,12,10,19]).

If M is measurable set with finite μ-measure, and if $f : M \to \mathbb{R}$ is a measurable function, the weighted rearrangement $f^\star : M^\star \to [0, +\infty[$ is uniquely defined by the following condition

$$\left\{x \in M^\star : f^\star(x) > t\right\} = \{x \in M : |f(x)| > t\}^\star \quad \forall t \geq 0. \tag{1.10}$$

This means that the super level sets of $f^\star$ are half-balls centered at the origin, having the same μ-measure of the corresponding super level sets of $|f|$.

Let C_μ denote the μ-measure of $B_1 \cap \mathbb{R}^N_+$. Using Theorem 1.1, we obtain the following comparison result.

Theorem 1.2. *Let u be the weak solution to problem (1.8), and let w be the function*

$$w(x) = w^\star(x) = \frac{1}{C_\mu} \int_{|x|}^{r^\star} \left(\int_0^\rho f^\star(\sigma)\sigma^{N-1+k} \exp\left(c\sigma^2\right) d\sigma \right) \rho^{-N+1-k} \exp\left(-c\rho^2\right) d\rho,$$

which is the weak solution to the problem

$$\begin{cases} -div\left(x_N^k \exp\left(c\,|x|^2\right) \nabla w\right) = x_N^k \exp\left(c\,|x|^2\right) f^\star & in D^\star \\ w = 0 & on \partial D^\star \cap \mathbb{R}^N_+. \end{cases} \tag{1.11}$$

Then

$$u^\star(x) \leq w(x) \quad a.e.\ in\ D^\star, \tag{1.12}$$

and

$$\int_D |\nabla u|^q \, d\mu \leq \int_{D^\star} |\nabla w|^q \, d\mu, \ for\ all\ 0 < q \leq 2. \tag{1.13}$$

2. Weighted isoperimetric inequalities in a cone of $\mathbb{R}^N$

In this section we study isoperimetric problems with respect to measures, relative to cones in $\mathbb{R}^N$. Notice that such problems have been investigated for instance in [20–25]. Our aim is to characterize those measures for which an isoperimetric set is given by the intersection of a cone with the ball having center at the vertex of the cone.

We begin by fixing some notation that will be used throughout: ω_N is the N-dimensional Lebesgue measure of the unit ball in $\mathbb{R}^N$. For points $x \in \mathbb{R}^N - \{0\}$ we will often use N-dimensional polar coordinates (r, Θ), where $r = |x|$ and $\Theta = x|x|^{-1} \in \mathbb{S}^{N-1}$. ∇_Θ denotes the gradient on $\mathbb{S}^{N-1}$. By $\mathbb{S}^{N-1}_+$ we denote the half sphere,

$$\mathbb{S}^{N-1}_+ = \mathbb{S}^{N-1} \cap \mathbb{R}^N_+.$$

Consider the isoperimetric problem (1.3), where Ω is the cone defined in (1.1) and ν the measure given by (1.2).

The first result of this section says that, if the isoperimetric set of (1.3) is $B_R \cap \Omega$ for a suitable R, then the density of the measure $d\nu$ is a product of two functions A and B of the variables r and Θ, respectively.

Note that it has been proven in [26] that a smooth density on $\mathbb{R}^N$ is radial if and only if spheres about the origin are stationary for a given volume.

Theorem 2.1. *Consider Problem (1.3), with $\phi \in C^1(\Omega) \cap C(\overline{\Omega})$, $\phi(x) > 0$ for $x \in \Omega$. Suppose that $I_\nu(m) = P_\nu(B_R \cap \Omega)$ whenever $m = \nu(B_R \cap \Omega)$, for every $R > 0$. Then*

$$\phi = A(r)B(\Theta), \tag{2.1}$$

where $A \in C^1((0, +\infty)) \cap C([0, +\infty))$, $A(r) > 0$ if $r > 0$, and $B \in C^1(\omega)$, $B(\Theta) > 0$ for $\Theta \in \omega$. Moreover, if $\phi \in C^2(\Omega)$, then

$$\lambda(B, \omega) \geq N - 1 + r^2 \left[\frac{(A'(r))^2}{(A(r))^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0, \tag{2.2}$$

where

$$\lambda(B, \omega) := \inf \left\{ \frac{\int_\omega |\nabla_\Theta u|^2 B \, d\Theta}{\int_\omega u^2 B \, d\Theta} : u \in C^1(\omega), \int_\omega uB \, d\Theta = 0, \ u \neq 0 \right\}. \tag{2.3}$$

Remark 2.1. Observe that $\lambda(B,\omega)$ is the first nontrivial eigenvalue of the Neumann problem

$$\begin{cases} -\nabla_\Theta\,(B\nabla_\Theta u) = \lambda Bu & \text{in } \omega \\ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\omega \end{cases}$$

where $u \in W^{1,2}(\omega)$, and $\mathbf{n}$ is the exterior unit normal to $\partial\omega$.

Proof of Theorem 2.1. Let $R > 0$. For $\varepsilon \in \mathbb{R}$ we define the following measure-preserving perturbations G_ε from $B_R \cap \Omega$:

$$G_\varepsilon := \{(r,\Theta) : 0 < r < R + \varepsilon h(\Theta) + s(\varepsilon),\ \Theta \in \omega\}, \quad |\varepsilon| \le \varepsilon_0$$

where $h \in C^1(\overline{\omega})$, and s is to be chosen such that $s \in C^2([-\varepsilon_0,\varepsilon_0])$, $s(0) = 0$, and $\nu(G_\varepsilon) = \nu(B_R)$ for $|\varepsilon| \le \varepsilon_0$. Writing $\phi = \phi(r,\Theta)$, and

$$R^\varepsilon := R + \varepsilon h + s(\varepsilon),$$

we have, for $|\varepsilon| \le \varepsilon_0$,

$$\nu(G_\varepsilon) = \int_\omega \int_0^{R^\varepsilon} r^{N-1}\phi(r,\Theta)\,dr\,d\Theta = \nu(B_R) \tag{2.4}$$

and

$$P_\nu(G_\varepsilon,\Omega) = \int_\omega (R^\varepsilon)^{N-2}\phi(R^\varepsilon,\Theta)\sqrt{(R^\varepsilon)^2 + |\nabla_\Theta R^\varepsilon|^2}\,d\Theta \ge P_\nu(B_R\cap\Omega,\Omega). \tag{2.5}$$

Denote $s_1 := s'(0)$ and $s_2 := s''(0)$. Differentiating (2.4) gives

$$0 = \int_\omega \phi(R,\Theta)(h(\Theta) + s_1)\,d\Theta, \tag{2.6}$$

and

$$0 = \int_\omega ((N-1)\phi(R,\Theta) + R\phi_r(R,\Theta))\,(h(\Theta)+s_1)^2\,d\Theta + s_2 R\int_\omega \phi(R,\Theta)\,d\Theta. \tag{2.7}$$

Using (2.5), we get

$$\begin{cases} \left.\dfrac{\partial}{\partial\varepsilon}P_\nu(G_\varepsilon,\Omega)\right|_{\varepsilon=0} = 0 \\ \left.\dfrac{\partial^2}{\partial\varepsilon^2}P_\nu(G_\varepsilon,\Omega)\right|_{\varepsilon=0} \ge 0. \end{cases} \tag{2.8}$$

The first condition in (2.8) gives

$$\int_\omega ((N-1)\phi(R,\Theta) + R\phi_r(R,\Theta))\,(h(\Theta)+s_1)\,d\Theta = 0. \tag{2.9}$$

In other words, we have that $\int_\omega ((N-1)\phi + R\phi_r)v\,d\theta = 0$ for all functions $v \in C^1(\overline{\omega})$ satisfying $\int_\omega \phi v\,d\theta = 0$. Then the Fundamental Lemma in the Calculus of Variations tells us that there is a number $k(R) \in \mathbb{R}$ such that

$$\phi_r(R,\Theta) = k(R)\phi(R,\Theta) \quad \forall\Theta \in \omega. \tag{2.10}$$

Integrating this with respect to R implies (2.1). Hence (2.6) and (2.7) give

$$0 = \int_\omega B(\Theta)(h(\Theta)+s_1)\,d\Theta, \tag{2.11}$$

$$0 = \left\{\frac{N-1}{R} + \frac{A'(R)}{A(R)}\right\}\cdot\int_\omega B(\Theta)(h(\Theta)+s_1)^2\,d\Theta + s_2\int_\omega B(\Theta)\,d\Theta. \tag{2.12}$$

Next, assume that $\phi \in C^2(\Omega)$. Then, using (2.1) and the second condition in (2.8), a short computation shows that

$$0 \le \left\{(N-2)(N-1)R^{N-3}A(R) + 2(N-1)R^{N-2}(A'(R))^{N-1}A''(R)\right\}\cdot\int_\omega B(\Theta)(h(\Theta)+s_1)^2\,d\Theta$$
$$+ s_2\left\{(N-1)R^{N-2}A(R) + R^{N-1}A'(R)\right\}\int_\omega B(\Theta)\,d\Theta + R^{N-3}A(R)\int_\omega B(\Theta)\,|\nabla_\Theta(h(\Theta)) + s_1|^2\,d\Theta.$$

Together with (2.12) this implies

$$0 \leq \left\{ -(N-1)R^{N-3}A(R) - R^{N-1}\frac{A'^2(R)}{A(R)} + R^{N-1}A''(R) \right\} \cdot \int_\omega B(\Theta)(h(\Theta)+s_1)^2\,d\Theta$$
$$+ R^{N-3}A(R)\int_\omega B(\Theta)|\nabla_\Theta(h(\Theta)+s_1)|^2\,d\Theta.$$

This implies (2.2), in view of (2.11) and the definition of $\lambda(B,\omega)$. □

Remark 2.2. The value of $\lambda(B,\omega)$ is explicitly known in some special cases. For instance (see, e.g. [27]), if $B \equiv 1$, and $\omega = \mathbb{S}^{N-1}$, we have

$$\lambda(1, \mathbb{S}^{N-1}) = N-1, \tag{2.13}$$

the eigenvalue has multiplicity N, with corresponding eigenfunctions $u_i(x) = x_i$, $(i = 1, \ldots, N)$, so that (2.2) reads as

$$A'^2 \leq A''(r)A(r), \tag{2.14}$$

or equivalently, A is log-convex, that is,

$$A(r) = e^{g(r)},$$

with a convex function g. It has been conjectured in [8], Conjecture 3.12, that for weights $\phi = A(r)$, with log-convex A, balls B_R, $(R > 0)$, solve the isoperimetric problem in $\mathbb{R}^N$.

Some partial answers to this conjecture are given in [28] and [29], and numerical evidence are provided in [26].

It is interesting to note that Theorem 1.1, whose proof will be the object of the next section, and Theorem 2.1 imply the following result.

Proposition 2.1. *Let $k \geq 0$, and*

$$B = B_k(\Theta) = \left(\frac{x_N}{|x|}\right)^k, \quad (x \in \mathbb{S}^{N-1}_+). \tag{2.15}$$

Then

$$\lambda(B_k, \mathbb{S}^{N-1}_+) = N-1+k, \tag{2.16}$$

with corresponding eigenfunctions

$$u_i = x_i, \quad (i = 1, \ldots, N-1). \tag{2.17}$$

Proof. Let u_i be given by (2.17). Theorems 1.1 and 2.1 imply that (2.2) holds, with $\omega = \mathbb{S}^{N-1}_+$, $A(r) = r^k e^{cr^2}$, $(c \geq 0)$, and $B(\Theta) = B_k(\Theta)$. Hence $\lambda(B_k, \mathbb{S}^{N-1}_+) \geq N-1+k-2cr^2$ for all $r > 0$, which implies that $\lambda(B_k, \mathbb{S}^{N-1}_+) \geq N-1+k$. The assertion follows from the identities

$$\int_{\mathbb{S}^{N-1}_+} |\nabla_\Theta u_i|^2 B_k\,d\Theta = (N-1+k)\int_{\mathbb{S}^{N-1}_+} (u_i)^2 B_k\,d\Theta, \quad \text{and}$$
$$\int_{\mathbb{S}^{N-1}_+} u_i B_k\,d\Theta = 0, \quad (i = 1, \ldots, N-1). \quad \Box$$

The next result gives the sharp constant in a weighted Hardy inequality with respect to the measure $x_N^k|x|^m\,dx$ in the quarter space $\{x_1 > 0, x_N > 0\}$ (for related results in half spaces, see e.g., [30–34]).

First we introduce some notation. Let D be an open set in $\mathbb{R}^N_+$, and ν a measure given by $d\nu = \phi(x)dx$, where $\phi \in L^\infty_{\text{loc}}(\mathbb{R}^N_+)$ and $\phi(x) > 0$. The weighted Hölder space $L^2(D, d\nu)$ is the set of all measurable functions $u : D \to \mathbb{R}$ such that $\int_D u^2\,d\nu < +\infty$, and the weighted Sobolev space $W^{1,2}(D, d\nu)$ is the set of functions $u \in L^2(D, d\nu)$ that possess weak partial derivatives $u_{x_i} \in L^2(D, d\nu)$, $(i = 1, \ldots, N)$. Norms in these spaces are given respectively by

$$\|u\|_{L^2(D,d\nu)} := \left(\int_D u^2\,d\nu\right)^{1/2},$$

and

$$\|u\|_{W^{1,2}(D,d\nu)} := \left(\int_D (|u^2 + |\nabla u|^2)\,d\nu\right)^{1/2}.$$

Definition 2.1. Let X be the set of all functions $u \in C^1(\overline{D})$ that vanish in a neighborhood of $\partial D \setminus \{x_N = 0\}$. Then let $V^2(D, d\nu)$ be the closure of X in the norm of $W^{1,2}(D, d\nu)$.

Next, let

$$Q := \{x \in \mathbb{R}^N : x_1 > 0,\ x_N > 0\}, \tag{2.18}$$

and specify

$$d\nu := x_N^k |x|^m \, dx, \tag{2.19}$$

where $k \geq 0$ and $m \in \mathbb{R}$.

Theorem 2.2. *With Q and ν given by (2.18) and (2.19) respectively, we have*

$$\int_Q |\nabla u|^2 \, d\nu \geq C(k, m) \int_Q \frac{u^2}{|x|^2} \, d\nu, \tag{2.20}$$

for all $u \in V^2(Q, d\nu)$, where

$$C(k, m) = \left(\frac{N+m+k-2}{2}\right)^2 + N + k - 1 = \left(\frac{N+m+k}{2}\right)^2 - m. \tag{2.21}$$

The constant $C(k, m)$ in (2.20) is sharp, and is not attained for any nontrivial function u.

Proof. We proceed as in [33, proof of Proposition 4.1].
Extend u to an odd function onto $\mathbb{R}_+^N$ by setting $u(-x_1, x_2, \ldots, x_N) := -u(x)$, $(x \in Q)$. Writing $u = u(r, \Theta)$ and $B_k(\Theta) = w(x) = x_N^k |x|^{-k}$, we have for a.e. $r > 0$,

$$\int_{\mathbb{S}_+^{N-1}} u(r, \Theta) B_k(\Theta) \, d\Theta = 0,$$

and thus by (2.16),

$$\int_{\mathbb{S}_+^{N-1}} |\nabla_\Theta u(r, \Theta)|^2 B_k(\Theta) \, d\Theta \geq (N + k - 1) \int_{\mathbb{S}_+^{N-1}} [u(r, \Theta)]^2 B_k(\Theta) \, d\Theta. \tag{2.22}$$

Further, the one-dimensional Hardy inequality (see [35]) tells us that for a.e. $\Theta \in \mathbb{S}_+^{N-1}$,

$$\int_0^{+\infty} r^{N+m+k-1} [u_r(r, \Theta)]^2 \, dr \geq \left(\frac{N+m+k-2}{2}\right)^2 \int_0^{+\infty} r^{N+m+k-3} [u(r, \Theta)]^2 \, dr. \tag{2.23}$$

Integrating (2.22) and (2.23) gives

$$\begin{aligned}
\int_{\mathbb{R}_+^N} |\nabla u|^2 \, d\nu &= \int_0^{+\infty} \int_{\mathbb{S}_+^{N-1}} \left([u_r]^2 + r^{-2} |\nabla_\Theta u|^2\right) r^{N-1+m+k} B_k \, d\Theta \, dr \\
&\geq \left[\left(\frac{N+m+k-2}{2}\right)^2 + N + k - 1\right] \int_0^{\infty} \int_{\mathbb{S}_+^{N-1}} u^2 r^{N+m+k-3} B_k \, d\Theta \, dr \\
&= C(k, m) \int_{\mathbb{R}_+^N} \frac{u^2}{|x|^2} \, d\nu.
\end{aligned}$$

The constant $C(k, m)$ is not attained since the constant is not attained in the one-dimensional Hardy inequality. Moreover, the exactness of $C(k, m)$ follows in a standard manner by considering functions of the form $u = u_n = x_1 |x|^{(-N-m-k)/2} \psi_n(|x|)$, $(n \in \mathbb{N})$, where $\psi_n \in C_0^\infty((0, +\infty))$, $0 \leq \psi_n \leq 1$, $|\psi_n'| \leq 4/n$, $\psi_n(t) = 0$ for $t \in (0, (1/n)] \cup [2n, +\infty)$, and $\psi_n(t) = 1$ for $t \in [(2/n), n]$, and then passing to the limit $n \to \infty$. The details are left to the reader. □

Theorem 2.1 has some further consequences when the cone Ω contains the wedge

$$W_+ := \{x = (x_1, \ldots, x_N) : x_i > 0,\ i = 1, \ldots, N\},$$

and if

$$\phi(x) = \prod_{i=1}^N \phi_i(x_i), \tag{2.24}$$

for some smooth functions ϕ_i, $(i = 1, \ldots, N)$.

In the following, let

$$\omega_+ := W_+ \cap \mathbb{S}^{N-1}.$$

We first show

Lemma 2.1. *Assume that $\phi \in C^2(W_+)$ satisfies (2.1) and (2.24), where $A, \phi_i \in C^2((0,+\infty)) \cap C([0,+\infty))$, $B \in C^2(\omega_+) \cap C(\overline{\omega_+})$, $\phi_i(x_i) > 0$ for $x_i > 0$, $(i = 1, \dots, N)$, $A(r) > 0$ for $r > 0$, and $B(\Theta) > 0$ for $\Theta \in \omega_+$. Then*

$$\phi(x) = a \prod_{i=1}^{N} x_i^{k_i} e^{c|x|^2}, \quad x \in W_+, \tag{2.25}$$

where $a > 0$, $k_i \geq 0$, $(i = 1, \dots, N)$, and $c \in \mathbb{R}$.

Proof. Differentiating the equation $\log[A(r)B(\Theta)] = \log[\prod_{i=1}^{N} \phi_i(x_i)]$ with respect to r gives

$$\frac{rA'(r)}{A(r)} = \sum_{i=1}^{N} \frac{x_i \phi_i'(x_i)}{\phi_i(x_i)}.$$

Differentiating this with respect to x_i yields

$$\frac{A'(r)}{rA(r)} + \frac{A''(r)}{A(r)} - \frac{(A'(r))^2}{(A(r))^2} = \frac{\phi_i'(x_i)}{x_i\phi_i(x_i)} + \frac{\phi_i''(x_i)}{\phi_i(x_i)} - \frac{(\phi_i'(x_i))^2}{(\phi_i(x_i))^2} = 4c, \quad (i = 1, \dots, N),$$

for some number $c \in \mathbb{R}$. In other words,

$$\frac{d}{dx_i}\left\{\frac{x_i\phi_i'(x_i)}{\phi_i(x_i)}\right\} = 4cx_i, \quad (i = 1, \dots, N).$$

Integrating this and dividing by x_i gives

$$\frac{\phi_i'(x_i)}{\phi_i(x_i)} = 2cx_i + \frac{k_i}{x_i}, \quad (i = 1, \dots, N),$$

for some numbers $k_i \in \mathbb{R}$, $(i = 1, \dots, N)$. Then another integration leads to

$$\log[\phi_i(x_i)] = b_i + k_i \log x_i + c(x_i)^2, \quad (b_i \in \mathbb{R}),$$

that is,

$$\phi_i(x_i) = a_i x_i^{k_i} e^{c(x_i)^2},$$

where $a_i = e^{b_i}$, $(i = 1, \dots, N)$. Since $\phi_i \in C([0, +\infty))$, and $\phi_i(x_i) > 0$ for $x_i > 0$, we have $a_i > 0$, and $k_i \geq 0$, $(i = 1, \dots, N)$. Now (2.25) follows with $a = \prod_{i=1}^{N} a_i$. □

As pointed out in the Introduction, we can specify the expression of the density ϕ of the measure, when the cone Ω is $\mathbb{R}^N_+$ and ϕ is factorized.

Theorem 2.3. *Assume $\Omega = \mathbb{R}^N_+$ and consider Problem (1.3), where $\phi \in C^1(\mathbb{R}^N_+) \cap C(\overline{\mathbb{R}^N_+})$, and satisfies (2.24), for some functions $\phi_i \in C^2(\mathbb{R})$, $\phi_i(t) > 0$ for $t \in \mathbb{R}$, $(i = 1, \dots, N-1)$, and $\phi_N \in C^2((0, +\infty)) \cap C([0, \infty))$, $\phi_N(t) > 0$ for $t > 0$. Suppose that $I_\nu(m) = P_\nu(B_R \cap \mathbb{R}^N_+, \mathbb{R}^N_+)$ for $m = \nu(B_R \cap \mathbb{R}^N_+)$. Then*

$$\phi(x) = a x_N^k e^{c|x|^2}, \tag{2.26}$$

for some numbers $a > 0$, $k \geq 0$ and $c \geq 0$.

Proof. By Theorem 2.1 we have $\phi = A(r)B(\Theta)$ with smooth positive functions A and B, and

$$\lambda(B, \mathbb{S}^{N-1}_+) \geq N - 1 + r^2\left[\frac{(A')^2}{A(r)^2} - \frac{A''(r)}{A(r)}\right] \quad \forall r > 0. \tag{2.27}$$

Then, Lemma 2.1 shows that ϕ satisfies (2.25). Since $\varphi(x) > 0$ whenever $x_N > 0$, and $x_i = 0$, for some $i \in \{1, \dots, N-1\}$, it follows that we must have $k_i = 0$, $(i = 1, \dots, N-1)$. This proves (2.26), for some numbers $a > 0$, $k \geq 0$ and $c \in \mathbb{R}$. Hence, $B(\Theta) = [x_N|x|^{-1}]^k$ and $A(r) = ar^k e^{cr^2}$. Therefore (2.16) and (2.27) imply that

$$N - 1 + k \geq N - 1 + k - 2cr^2 \quad \forall r > 0.$$

Hence we must have $c \geq 0$. □

We end this section by analyzing the case where the cone Ω is $\mathbb{R}^N \setminus \{0\}$.

Theorem 2.4. *Assume* $\Omega = \mathbb{R}^N \setminus \{0\}$ *and consider Problem (1.3), with* $\phi \in C^2(\mathbb{R}^N \setminus \{0\}) \cap C(\mathbb{R}^N)$, $\phi(x) > 0$ *for* $x \neq 0$, *and satisfies (2.24), where* $\phi_i \in C^2(\mathbb{R} \setminus \{0\}) \cap C(\mathbb{R})$, *and* $\phi_i(t) > 0$ *for* $t \neq 0$, $(i = 1, \dots, N)$. *Suppose that* $I_\nu(m) = P_\nu(B_R)$ *for* $m = \nu(B_R)$. *Then*

$$\phi(x) = ae^{c|x|^2}, \tag{2.28}$$

for some numbers $a > 0$, *and* $c \geq 0$.

Proof. By Theorem 2.1 we have $\phi = A(r)B(\Theta)$ with smooth positive functions A and B, and

$$\lambda(B, \mathbb{S}^{N-1}) \geq N - 1 + r^2 \left[\frac{A'(r)^2}{A(r)^2} - \frac{A''(r)}{A(r)} \right] \quad \forall r > 0. \tag{2.29}$$

Then, Lemma 2.1 shows that ϕ satisfies (2.25). Since $\varphi(x) > 0$ whenever $x \neq 0$ and $x_i = 0$, for some $i \in \{1, \dots, N\}$, it follows that $k_i = 0$, $(i = 1, \dots, N)$. This proves (2.28), for some numbers $a > 0$, and $c \in \mathbb{R}$, that is, $B(\Theta) \equiv 1$ and $A(r) = ae^{cr^2}$. Hence, (2.13) and (2.29) imply that A is log-convex, that is, we must have $c \geq 0$. □

3. A Dido's problem

In this section we provide the proof of Theorem 1.1. As pointed out in the Introduction, we have to find the set having minimum μ-perimeter among all the subsets of $\mathbb{R}^N_+$ having prescribed μ-measure, where μ is the measure defined in (1.7). In order to face such a problem we first show a simple inequality for measures defined on the real line related to $d\mu$. Then the isoperimetric problem is addressed in the plane: the one-dimensional results allows one to restrict the search of optimal sets to the ones which are starlike with respect to the origin. Finally Theorem 1.1 is achieved in its full generality.

3.1. Dido's problem on the real line

Let $\mathbb{R}_+ = (0, +\infty)$. The following isoperimetric inequality holds.

Proposition 3.1. *Let* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *be a nondecreasing continuous function,* $d\nu = \phi(x)dx$ *and* M *be a measurable subset of* $\mathbb{R}_+$ *with* $\nu(M) < +\infty$. *Then*

$$P_\nu(M) \geq P_\nu(S(M)), \tag{3.1}$$

where $S(M)$ *denotes the interval* $(0, d)$, *with* $d \geq 0$ *chosen such that* $\nu(M) = \nu(S(M))$.

Proof. First assume that M is of the form

$$M = \cup_{j=1}^k \left(a_j, b_j\right), \tag{3.2}$$

with

$$0 \leq a_j < a_{j+1}, \quad a_j < b_j, \quad b_j < b_{j+1} < +\infty,$$

for all $j \in \{1, \dots, k-1\}$. By the properties of the weight function ϕ we have that $b_k \geq d$ and hence

$$P_\nu(M) = \sum_{j=1}^k \left[\phi(a_j) + \phi(b_j)\right] \geq \phi(0) + \phi(d) = P_\nu(S(M)). \tag{3.3}$$

Next let M be measurable and $\nu(M) < +\infty$. By the basic properties of the perimeter, there exists a sequence of sets $\{M_n\}$ of the form (3.2) such that $\lim_{n\to+\infty} \nu(M \Delta M_n) = 0$ and $\lim_{n\to+\infty} P_\nu(M_n) = P_\nu(M)$. The first limit implies that also $\lim_{n\to+\infty} P_\nu(S(M_n)) = P_\nu(S(M))$, so that the assertion follows from inequality (3.3). □

3.2. Dido's problem in two dimensions

In our study of the measure $d\mu$, an important role will be played by the following isoperimetric theorem (see [2,8]) relative to the measure

$$d\tau = \exp(c\,|x|^2)dx, \quad x \in \mathbb{R}^m, \text{ with } m \geq 1 \text{ and } c \geq 0.$$

Theorem 3.1. *If M is any measurable subset of $\mathbb{R}^N$ and $M^\star$ is the ball of $\mathbb{R}^N$ centered at the origin having the same τ-measure of M, then*

$$P_\tau(M) \geq P_\tau(M^\star). \tag{3.4}$$

We write (x, y) for points in $\mathbb{R}^2$, and we consider in $\mathbb{R}^2_+$ the measure

$$d\mu = y^k \exp\left(c(x^2 + y^2)\right) dx\, dy,$$

where $c \geq 0$ and $k \geq 0$. If M is a measurable subset of $\mathbb{R}^2_+$, given any number $m > 0$, the isoperimetric problem on $\mathbb{R}^2_+$ reads as:

$$I_\mu(m) := \inf\{P_\mu(M),\ \text{with}\ M : \mu(M) = m\}. \tag{3.5}$$

The following result holds true.

Theorem 3.2. *Let $m > 0$. Then $I_\mu(m)$ is attained for the half-disk $B_r \cap \mathbb{R}^2_+$, centered at zero, having μ-measure m. Equivalently there exists $r > 0$ such that*

$$I_\mu(m) = P_\mu(B_r \cap \mathbb{R}^2_+) = \exp\left(cr^2\right) r^{k+1} \int_0^\pi \sin^k\theta\, d\theta = B\left(\frac{k+1}{2}, \frac{1}{2}\right) \exp\left(cr^2\right) r^{k+1}, \tag{3.6}$$

where B denotes the Beta function.

Proof. If $k = 0$, and $c = 0$ (unweighted case), the result is well-known. Further, if $c > 0$ and $k = 0$, that is, $d\mu = e^{c(x^2+y^2)}\, dx\, dy$, the result follows from Theorem 3.1 via reflexion about the x-axis. Finally, the result has been shown in the case $c = 0$ and $k > 0$ by Maderna and Salsa, [10], (see also [36]).

Therefore we may restrict ourselves to the case that both c and k are positive.

Our proof requires some technical effort, mainly due to the degeneracy of the measure on the x-axis. The strategy is as follows: First we use symmetrization arguments in order to reduce the isoperimetric problem to sets which are starlike w.r.t. the origin (*Step* 1). Then we obtain some a priori estimates for a minimizing sequence (*Step* 2). This allows us to show that a (starlike) minimizer exists (*Step* 3), which is also bounded (*Step* 4) and smooth (*Step* 5). In *Step* 6 we evaluate the second variation of the Perimeter functional, and we show that the minimizer is a half-disk centered at the origin.

Throughout our proof, C will denote a generic constant, which may vary from line to line.

Step 1: Symmetrization

Our aim is to simplify the isoperimetric problem using Steiner symmetrization in two directions. This method has already been employed in the case $c = 0$ (see [10,36]).

Let $\{D_n\} \subset \mathbb{R}^2_+$ be a minimizing sequence for problem (3.5), i.e.

$$\mu(D_n) = m \quad \forall n \in \mathbb{N}\ \text{and}\ \lim_{n\to+\infty} P_\mu(D_n) = I_\mu(m),$$

where, without loss of generality, we may assume that the sets D_n are smooth.

Let D be a smooth set of $\mathbb{R}^2_+$. We denote by $S_x(D)$ and $S_y(D)$ the Steiner symmetrization in the x-direction, with respect to the measure $d\mu_x = e^{cx^2} dx$, and the Steiner symmetrization in the y-direction, with respect to the measure $d\mu_y = e^{cy^2} y^k\, dy$, of D, respectively.

More precisely, $S_x(D)$ is the subset of $\mathbb{R}^2_+$ whose cross sections parallel to the x-axis are open intervals centered at the y-axis, and such that their μ_x-lengths are equal to those of the corresponding cross sections of D.

The set $S_y(D)$ is defined in a similar way: its cross sections parallel to the y-axis are open intervals with an endpoint lying on the x-axis, and such that their μ_y-lengths are equal to those of the corresponding cross sections of D.

Now consider the sequence of sets $M_n = S_y(S_x(D_n))$. By Proposition 3.1 and Theorem 3.1, we have that $P_\mu(S_y(S_x(D_n))) \leq P_\mu(D_n)$ and, by Cavalieri's principle, $\mu(S_y(S_x(D_n))) = \mu(D_n)$. Therefore $\{M_n\}$ is still a minimizing sequence for (3.5). On one hand, the sets M_n can lose regularity under symmetrization: the symmetrized sets are not more than locally Lipschitz continuous, in general. On the other hand, they acquire some nice geometrical property: they are all starlike with respect to the origin. Thus, introducing polar coordinates (r, θ) as $x = r\cos\theta$ and $y = r\sin\theta$, we have

$$M_n = \{(r, \theta) : 0 < r < \rho_n(\theta),\ \theta \in (0, \pi)\}, \quad \forall n \in \mathbb{N}, \tag{3.7}$$

for some functions $\rho_n(\theta) : (0,\pi) \to (0,+\infty)$. Note that, defining $\rho_n(0) := \lim_{\theta\to 0^+} \rho_n(\theta) =: \rho_n(\pi)$, and $\rho_n(\pi/2) := \lim_{\theta\to\pi/2^-} \rho_n(\theta)$, we have also have $\rho_n \in C([0,\pi])$. Then

(i) the functions $\rho_n(\theta)$ are locally Lipschitz in $(0,\pi/2)$;
(ii) $\rho_n(\theta) = \rho_n(\pi-\theta),\ \forall n\in\mathbb{N},\ \forall\theta\in(0,\pi)$;
(iii) the functions $x_n(\theta) := \rho_n(\theta)\cos\theta$ and $y_n(\theta) := \rho_n(\theta)\sin\theta$ are nonincreasing and nondecreasing, respectively, on $(0,\pi/2)$.

Hence we may assume that the minimizing sequence is of the form (3.7), with conditions (i)–(iii) in force. Under these conditions, the set M_n, its μ-measure and μ-perimeter are uniquely determined by the function $\rho_n(\theta)$. More precisely, setting

$$z := \sin^k\theta, \quad \theta\in[0,\pi],$$
$$F(r) := \int_0^r e^{ct^2} t^{k+1}\,dt, \quad \text{and}$$
$$G(r,p) := e^{cr^2} r^k\sqrt{r^2+p^2}, \quad r>0,\ p\in\mathbb{R},$$

we find that

$$\mu(M_n) = \int_0^\pi F(\rho_n) z\,d\theta =: \mu(\rho_n), \quad \text{and}$$
$$P_\mu(M_n) = \int_0^\pi G(\rho_n,\rho_n') z\,d\theta =: P_\mu(\rho_n).$$

With this notation, the isoperimetric problem (3.5) now reads as

$$\text{Minimize} \quad P_\mu(\rho) \text{over}$$
$$K := \{\rho : (0,\pi/2)\cup(\pi/2,\pi) \to (0,+\infty) :\ \rho \text{ satisfies (i)-(iii) and } \mu(\rho)=m\}. \tag{3.8}$$

Step 2: Some estimates

Next we will obtain some uniform estimates for the minimizing sequence $\{\rho_n\}$ of problem (3.8).

Condition (iii) implies

$$-\rho_n(\theta)\cot\theta \le \rho_n'(\theta) \le \rho_n(\theta)\tan\theta \ \text{ a.e. on } \ (0,\pi/2),\ n\in\mathbb{N}. \tag{3.9}$$

Set

$$y_n^0 := \sup_{\theta\in(0,\pi/2)} y_n(\theta) = y_n(\pi/2) = \rho_n(\pi/2).$$

We claim that

$$\sup_{n\in\mathbb{N}} y_n^0 =: y^0 < +\infty. \tag{3.10}$$

Indeed, since $\{P_\mu(\rho_n)\}$ is a bounded sequence, we obtain for every $n\in\mathbb{N}$,

$$\begin{aligned} C &\ge P_\mu(\rho_n) \\ &= 2\int_0^{\pi/2} e^{c(x_n^2(\theta)+y_n^2(\theta))} y_n^k(\theta)\sqrt{(x_n'(\theta))^2+(y_n'(\theta))^2}\,d\theta \\ &\ge 2\int_0^{\pi/2} e^{cy_n^2(\theta)} y_n^k(\theta) y_n'(\theta)d\theta = 2\int_0^{y_n^0} e^{ct^2} t^k dt, \end{aligned}$$

and (3.10) follows.

From (3.9) and (3.10), we further deduce that for every $\theta\in(0,\pi)$,

$$\rho_n(\theta) = \frac{y_n(\theta)}{\sin\theta} \le \frac{y_n(\pi/2)}{\sin\theta} \le \frac{y^0}{\sin\theta} \quad \forall n\in\mathbb{N}. \tag{3.11}$$

Conditions (3.11) and (3.9) imply that for every $\delta\in(0,\pi/4)$ there is a number $d_\delta>0$ such that

$$\sup_{\theta\in(\delta,\pi/2-\delta)} \left\{\rho_n(\theta), \left|\rho_n'(\theta)\right|\right\} \le d_\delta. \tag{3.12}$$

Next we claim:

There exists a number $d_1 > 0$, such that

$$\rho_n(\theta) \geq d_1 \quad \forall \theta \in (0, \pi), \quad \text{and } \forall n \in \mathbb{N}. \tag{3.13}$$

Assume (3.13) was not true. Then the fact that $x_n(\theta)$ and $y_n(\theta)$ are nonincreasing, respectively nondecreasing, $\forall n \in \mathbb{N}$, means that there is a subsequence, still labelled as $\{\rho_n\}$, such that $\lim_{n\to\infty} \rho_n(\pi/4) = 0$. Set $\delta_n := \rho_n(\pi/4)/\sqrt{2}$ and note that $x_n(\pi/4) = y_n(\pi/4) = \delta_n$. In view of (3.11) we have that

$$\lim_{n\to\infty} \mu(M_n \cap \{|x| < \delta_n\}) = 0.$$

Since $\mu(\rho_n) = m$, this implies that there is a number $d_2 > 0$, such that for all $n \in \mathbb{N}$,

$$\begin{aligned} d_2 &\leq \mu(M_n \cap \{x > \delta_n\}) = -\int_0^{\pi/4} x_n'(\theta) e^{cx_n^2(\theta)} \int_0^{y_n(\theta)} e^{ct^2} t^{k+1}\, dt\, d\theta \\ &\leq -(\delta_n)^2 \int_0^{\pi/4} x_n'(\theta) e^{c(x_n^2(\theta)+y_n^2(\theta))} y_n^k(\theta)\, d\theta. \end{aligned} \tag{3.14}$$

On the other hand, the sequence $\{P_\mu(\rho_n)\}$ is bounded, so that

$$\begin{aligned} C &\geq P_\mu(\rho_n) \\ &\geq \int_0^{\pi/4} e^{c(x_n^2(\theta)+y_n^2(\theta))} y_n^k(\theta) \sqrt{(x_n'(\theta))^2 + (y_n'(\theta))^2}\, d\theta \\ &\geq -\int_0^{\pi/4} x_n'(\theta) e^{c(x_n^2(\theta)+y_n^2(\theta))} y_n^k(\theta)\, d\theta. \end{aligned} \tag{3.15}$$

Hence we obtain $d_2 \leq \delta_n^2 C$ for all $n \in \mathbb{N}$, which is a contradiction.

Next, we claim that there is a number $d_3 > 0$ such that holds for every $\theta \in (0, \pi/2)$ and for all $n \in \mathbb{N}$, there holds

$$y_n^k(\theta) \int_0^{x_n(\theta)} e^{ct^2}\, dt \leq d_3. \tag{3.16}$$

Consider the set

$$\widetilde{M}_n(\theta) := \{(x, y) \in M_n : y \leq y_n(\theta)\}.$$

It is easy to verify that

$$\begin{aligned} \frac{1}{2}\left(P_\mu(M_n) - P_\mu(\widetilde{M}_n(\theta))\right) &= \int_\theta^{\pi/2} e^{c(x_n^2(\tau)+y_n^2(\tau))} y_n(\tau)^k \sqrt{(x_n'(\tau))^2 + (y_n'(\tau))^2}\, d\tau - \int_0^{x_n(\theta)} e^{c(t^2+y_n^2(\theta))} y_n^k(\theta)\, dt \\ &\geq \int_\theta^{\pi/2} (-x_n'(\theta)) e^{cx_n^2(\tau)} \left(e^{y_n^2(\tau)} y_n^k(\tau) - e^{y_n^2(\theta)} y_n^k(\theta)\right) d\tau \geq 0. \end{aligned}$$

Hence

$$C \geq P_\mu(M_n) \geq P_\mu(\widetilde{M}_n(\theta)) \geq 2 y_n^k(\theta) e^{cy_n^2(\theta)} \int_0^{x_n(\theta)} e^{ct^2}\, dt, \tag{3.17}$$

and (3.16) follows.

Below we will frequently make use of the following limit, which holds for all $\alpha > -1$,

$$\lim_{z\to+\infty} \frac{\int_0^z e^{ct^2} t^\alpha\, dt}{e^{cz^2} z^{\alpha-1}} = \frac{1}{2c}. \tag{3.18}$$

In view of (3.18) with $\alpha = 0$, and (3.17), and since $x_n(\theta) \geq d/\sqrt{2}$ for $\theta \in (0, \pi/4)$, we obtain

$$C \geq y_n^k(\theta) e^{cy_n^2(\theta)} \frac{e^{cx_n^2(\theta)}}{x_n(\theta)}, \quad \forall \theta \in (0, \pi/4). \tag{3.19}$$

Since $y_n(\theta) \geq (1/2)\theta \rho_n(\theta)$ for $\theta \in (0, \pi/4)$, and $x_n(\theta) \leq \rho_n(\theta)$, we further deduce from (3.19),

$$C \geq \rho_n^{k-1}(\theta) \theta^k e^{c\rho_n^2(\theta)}, \quad \forall \theta \in (0, \pi/4). \tag{3.20}$$

Now recall (3.13), and $\lim_{z\to+\infty} e^{cz^2/2}z^{k-1} = +\infty$. Hence (3.20) shows that there is a number $d_4 > 0$ such that for all $n \in \mathbb{N}$,

$$\rho_n(\theta) \le \sqrt{d_4 - \frac{2k}{c}\ln\theta}, \quad \forall\theta \in (0, \pi/4). \tag{3.21}$$

Finally we show:

$$\text{For every } \epsilon \in (0, m) \text{ there is a } \delta \in (0, \pi/2), \text{ such that} \\ \mu(M_n \cap \{\delta < \theta < \pi - \delta\}) > m - \epsilon, \quad \forall n \in \mathbb{N}. \tag{3.22}$$

Indeed, (3.18), (3.20) and (3.21) with $\alpha = k + 1$, show that

$$\begin{aligned} \mu(M \cap \{0 < \theta < s\}) &= \int_0^s \sin^k\theta \int_0^{\rho_n(\theta)} e^{ct^2} t^{k+1}\,dt\,d\theta \\ &\le C\int_0^s \theta^k e^{c\rho_n^2(\theta)}\rho_n^k(\theta)\,d\theta \\ &\le C\int_0^s \sqrt{d_4 - \frac{2k}{c}\ln\theta}\,d\theta \to 0, \quad \text{as } s \to 0. \end{aligned} \tag{3.23}$$

Now the claim (3.22) follows from the uniform estimate (3.23) and from the fact that for every $s \in (0, \pi/2)$,

$$m/2 = \mu(M_n \cap \{0 < \theta < s\}) + \mu(M_n \cap \{s < \theta < \pi/2\}).$$

Step 3: The minimum is achieved.

In this step we show that a minimizer of problem (3.8) exists.

In view of the properties (i)–(iii), (3.9), and the estimates (3.11)–(3.13) and (3.21) there exists a function $\rho : (0, \pi/2) \cup (\pi/2, \pi) \to [0, +\infty)$ which is locally Lipschitz continuous, and a subsequence, still denoted by $\{\rho_n\}$, such that

$$\rho_n \to \rho \text{ uniformly on compact subsets of } (0, \pi/2), \tag{3.24}$$

$$\rho(\theta) = \rho(\pi - \theta) \quad \forall\theta \in (0, \pi/2), \tag{3.25}$$

$$-\rho(\theta)\cot\theta \le \rho'(\theta) \le \rho(\theta)\tan\theta \quad \text{a.e. on } (0, \pi/2), \tag{3.26}$$

$$\rho(\theta) \le \frac{y^0}{\sin\theta} \quad \forall\theta \in (0, \pi/2), \tag{3.27}$$

$$\rho(\theta) \ge d_1 \quad \forall\theta \in (0, \pi/2), \tag{3.28}$$

$$\sup_{\theta\in(\delta,\pi/2-\delta)} \left\{\rho(\theta), \left|\rho'(\theta)\right|\right\} \le d_\delta, \quad \forall\delta \in (0, \pi/4), \tag{3.29}$$

$$\rho(\theta) \le \sqrt{d_4 - \frac{2k}{c}\ln\theta}, \quad \forall\theta \in (0, \pi/4). \tag{3.30}$$

Note, setting $x(\theta) := \rho(\theta)\cos\theta$, and $y(\theta) := \rho(\theta)\sin\theta$, condition (3.26) implies that the functions $x(\theta)$ and $y(\theta)$ are nonincreasing, respectively nondecreasing on $(0, \pi/2)$. Further, defining $\rho(\pi/2) := \lim_{\theta\to\pi/2}\rho(\theta)$, we see that $\rho \in C((0, \pi))$.

Let

$$M := \{(r, \theta) : 0 < r < \rho(\theta), \ \theta \in (0, \pi)\},$$

and $P_\mu(\rho) := P_\mu(M)$. We claim

$$\mu(M) = m. \tag{3.31}$$

Indeed, the estimate (3.22) shows

$$\text{For every } \epsilon \in (0, m) \text{ there is a } \delta \in (0, \pi/2), \text{ such that} \\ \mu(M \cap \{\delta < \theta < \pi - \delta\}) \ge m - \epsilon.$$

Since we also have $\mu(M) \le m$, (3.31) follows.

Finally, the lower semicontinuity of the perimeter shows that

$$I_\mu(m) = \lim_{n\to\infty} P_\mu(M_n) \ge P_\mu(M). \tag{3.32}$$

But $\rho \in K$, therefore $I_\mu(m) = P_\mu(M)$, and M is a minimizer.

Note that our weight function $\phi(x, y) := y^k e^{c(x^2+y^2)}$ is positive and $\phi \in C^\infty(\mathbb{R}^2_+)$. Due to a regularity result of Morgan [37], Corollary 3.7 and Remark 3.10, this implies that $\partial M \cap \mathbb{R}^2_+$ is a one-dimensional C^1-manifold that is locally analytic. In view of the symmetry of M this implies that ρ is differentiable at $\pi/2$, with $\lim_{\theta\to\pi/2} \rho'(\theta) = \rho'(\pi/2) = 0$. Using the properties (3.25)–(3.30) this implies that $\rho \in C^\infty((0, \pi))$.

Then standard Calculus of Variations (see [10]) shows that there is a number $\gamma \in \mathbb{R}$ – a Lagrangian multiplier – such that

$$-\frac{d}{d\theta}\left(G_p z\right) + G_r z = \gamma F' z \quad \text{on } (0, \pi). \tag{3.33}$$

Here, and in the following, the functions G, F and their derivatives are evaluated at (ρ, ρ').

Step 4 : The minimizer is bounded.

We will argue by contradiction, that is, we assume that ρ was unbounded. Then (3.26) would imply that

$$\lim_{t\to 0} \rho(t) = +\infty. \tag{3.34}$$

First we claim that (3.34) further means that there exists a sequence $t_n \to 0$ such that

$$-\rho'(t_n) \geq \rho^3(t_n). \tag{3.35}$$

Indeed, assume (3.35) was not true. Then there exists a number $t_0 > 0$ such that

$$-\rho'(t) < \rho^3(t) \quad \text{for } t \in (0, t_0). \tag{3.36}$$

By the estimate (3.30) we can find a number $t_1 \in (0, t_0)$ such that $-2t_1 + (\rho(t_1))^{-2} =: \delta_0 > 0$. Integrating (3.36) gives

$$\frac{1}{\rho^2(t)} > 2t - 2t_1 + \frac{1}{(\rho(t_1))^2} = \delta_0 + 2t \quad \forall t \in (0, t_1),$$

which implies that ρ is bounded, a contradiction. Hence (3.35) follows. Note that (3.35), together with our assumption (3.34), implies that

$$\lim_{n\to\infty} \rho(t_n)/\rho'(t_n) = 0. \tag{3.37}$$

Using the Euler equation (3.33), a short calculation shows that

$$\frac{d}{d\theta}\left(G - \rho' G_p - \gamma F\right) = \rho' G_p \frac{z'}{z}. \tag{3.38}$$

Integrating (3.38) on the interval $(t_n, \pi/2)$ gives

$$\begin{aligned}
&\gamma \int_0^{\rho(t_n)} e^{cs^2} s^{k+1} ds - e^{c(\rho(t_n))^2} (\rho(t_n))^{k+2} \left((\rho(t_n))^2 + (\rho'(t_n))^2\right)^{-1/2} \\
&\quad = -c_1 + \int_{t_n}^{\pi/2} e^{c(\rho(t))^2} (\rho(t))^k (\rho'(t))^2 \left((\rho(t))^2 + (\rho'(t))^2\right)^{-1/2} k \cot t \, dt,
\end{aligned} \tag{3.39}$$

where we have put

$$c_1 = (G - \rho' G_p - \gamma F)|_{\theta=\pi/2}.$$

In view of (3.34), (3.35), (3.37) and (3.18), with $\alpha = k + 1$, we find that

$$\lim_{n\to\infty} \frac{\gamma \int_0^{\rho(t_n)} e^{cs^2} s^{k+1} ds}{e^{c(\rho(t_n))^2} (\rho(t_n))^{k+2} \left((\rho(t_n))^2 + (\rho'(t_n))^2\right)^{-1/2}} = +\infty. \tag{3.40}$$

Hence the left-hand side of Eq. (3.39) tends to $+\infty$ as $n \to +\infty$. Using L'Hopital's rule, (3.34), (3.35) and (3.37), we obtain from (3.40),

$$\begin{aligned}
1 &= \lim_{n\to\infty} \frac{\gamma \int_0^{\rho(t_n)} e^{cs^2} s^{k+1} ds - e^{c(\rho(t_n))^2} (\rho(t_n))^{k+2} \left((\rho(t_n))^2 + (\rho'(t_n))^2\right)^{-1/2}}{-c_1 + \int_{t_n}^{\pi/2} e^{c(\rho(t))^2} (\rho(t))^k (\rho'(t))^2 \left((\rho(t))^2 + (\rho'(t))^2\right)^{-1/2} k \cot t \, dt} \\
&= \lim_{n\to\infty} \frac{\gamma \int_0^{\rho(t_n)} e^{cs^2} s^{k+1} ds}{\int_{t_n}^{\pi/2} e^{c(\rho(t))^2} (\rho(t))^k (\rho'(t))^2 \left((\rho(t))^2 + (\rho'(t))^2\right)^{-1/2} k \cot t \, dt} \\
&= \lim_{n\to\infty} \frac{\gamma \rho'(t_n) e^{c(\rho(t_n))^2} (\rho(t_n))^{k+1}}{-e^{c(\rho(t_n))^2} (\rho(t_n))^k (\rho'(t_n))^2 \left((\rho(t_n))^2 + (\rho'(t_n))^2\right)^{-1/2} k \cot t_n}
\end{aligned}$$

$$= \lim_{n\to\infty} \frac{\gamma \rho'(t_n) e^{c(\rho(t_n))^2}(\rho(t_n))^{k+1}}{e^{c(\rho(t_n))^2}(\rho(t_n))^k \rho'(t_n) k \cot t_n}$$
$$= \lim_{n\to\infty} \frac{\gamma \rho(t_n)}{k \cot t_n} = \lim_{n\to\infty} \frac{\gamma}{k} t_n \rho(t_n).$$

But the last limit is zero in view of (3.30), and we have obtained a contradiction. In other words, ρ is bounded on $(0, \pi)$.

Putting $\rho(0) := \lim_{t\to 0} \rho(t) =: \rho(\pi)$, we then have

$$\rho \in C([0, \pi]). \tag{3.41}$$

Step 5: ρ'' is bounded.

We will first need some integrability properties of the functions

$$G_r = e^{c\rho^2} \rho^k \left([2c\rho + (k/\rho)]\{\rho^2 + (\rho')^2\}^{1/2} + \rho\{\rho^2 + (\rho')^2\}^{-1/2}\right),$$
$$G_p = e^{c\rho^2} \rho^k \rho' \{\rho^2 + (\rho')^2\}^{-1/2},$$
$$F' = e^{c\rho^2} \rho^{k+1}.$$

By (3.28) and (3.41), G_p and F' are bounded on $(0, \pi)$. Moreover, since $P_\mu(\rho) < +\infty$, we also have $G_r z \in L^1((0, \pi))$. Integrating (3.33) between 0 and $t \in (0, \pi/2)$ gives

$$\begin{aligned} -G_p(\rho(t), \rho'(t))z(t) &= \int_0^t (\gamma F' - G_r) z \, d\theta \\ &= \int_0^t z e^{c\rho^2} \rho^k \left(\gamma\rho - (2c\rho + (k/\rho)) \left[\rho^2 + (\rho')^2\right]^{1/2} - \rho \left[\rho^2 + (\rho')^2\right]^{-1/2}\right) d\theta \\ &\le \gamma \int_0^t e^{c\rho^2} \rho^{k+1} z \, d\theta \le C \int_0^t z \, d\theta \le C t^{k+1}. \end{aligned} \tag{3.42}$$

On the other hand, if $\rho'(t) < 0$, then (3.28) and the boundedness of ρ show that

$$\begin{aligned} -G_p(\rho(t), \rho'(t))z(t) &= -e^{c(\rho(t))^2} (\rho(t))^k \rho'(t) \left[\rho^2(t) + (\rho')^2\right]^{-1/2} \sin^k t \\ &\ge -C\rho'(t) \left[C^2 + (\rho')^2\right]^{-1/2} t^k. \end{aligned} \tag{3.43}$$

Furthermore, the estimate (3.26) and the boundedness of ρ imply that there is a constant $d_5 > 0$ such that

$$\rho'(t) \le d_5 t \quad \forall t \in (0, \pi/2). \tag{3.44}$$

Now (3.43), (3.42) and (3.44) imply that

$$\rho'(t)/t \text{ is bounded on } (0, \pi/2). \tag{3.45}$$

In particular we have $\rho \in C^1([0, \pi])$ and $\rho'(0) = \rho'(\pi) = 0$.

Finally, using (3.33), a short calculation gives

$$\gamma\rho = \frac{-\rho^2 \rho'' + \rho(\rho')^2}{(\rho^2 + (\rho')^2)^{3/2}} - \frac{[(k/\rho) + 2c\rho](\rho')^2 + k\rho' \cot t}{(\rho^2 + (\rho')^2)^{1/2}}. \tag{3.46}$$

By (3.28), (3.41) and (3.45) this implies that

$$\rho'' \in L^\infty((0, \pi)). \tag{3.47}$$

Step 6: M is a half-disk.

Note first that the derivatives G_{rr}, G_{rp}, G_{pp} and F'' are bounded, in view of the properties (3.28), (3.41) and (3.45).

Since ρ is a minimizer of (3.8), the second variation of P_μ at ρ in K is nonnegative. This means that

$$0 \le \int_0^\pi \left(G_{rr}\kappa^2 + 2G_{rp}\kappa\kappa' + G_{pp}(\kappa'')^2 - \gamma F''\kappa^2\right) z \, d\theta, \tag{3.48}$$

for every $\kappa \in W^{1,2}((0, \pi))$ such that

$$\int_0^\pi F' \kappa z \, d\theta = 0. \tag{3.49}$$

Furthermore, dividing (3.33) by z and then differentiating yields

$$G_{rr}\rho' + G_{rp}\rho'' - \frac{d}{d\theta}\left(G_{rp}\rho' + G_{pp}\rho''\right) - \left(G_{pr}\rho' + G_{pp}\rho''\right)\frac{z'}{z} - G_p\left(\frac{z'}{z}\right)' = \gamma F''\rho' \quad \text{in } (0,\pi). \tag{3.50}$$

Multiplying (3.50) by $\rho' z$ and then integrating by parts, we obtain

$$\int_0^\pi G_p\rho'\left(\frac{z'}{z}\right)' z\,d\theta = \int_0^\pi \left(G_{rr}(\rho')^2 + 2G_{rp}\rho'\rho'' + G_{pp}(\rho'')^2 - \gamma F''(\rho')^2\right) z\,d\theta. \tag{3.51}$$

Note that we may use (3.48) with $\kappa = \rho' \in W^{1,\infty}((0,\pi))$. This shows that the right-hand side of (3.51) is nonnegative. On the other hand,

$$\int_0^\pi G_p\rho'\left(\frac{z'}{z}\right)' z\,d\theta = -k\int_0^\pi e^{c\rho^2}\rho^k\left(\rho^2 + (\rho')^2\right)^{-1/2}(\rho')^2 \sin^{k-2}\theta\,d\theta \le 0. \tag{3.52}$$

Hence

$$\int_0^\pi e^{c\rho^2}\rho^k\left(\rho^2 + (\rho')^2\right)^{-1/2}(\rho')^2 \sin^{k-2}\theta\,d\theta = 0, \tag{3.53}$$

which implies that $\rho' = 0$ in $[0,\pi]$. This means that ρ is constant in $[0,\pi]$, and the result follows. □

3.3. The N-dimensional case

Proof of Theorem 1.1. We proceed by induction over the dimension N. Note that the result for $N = 2$ is Theorem 3.2. Assume that the assertion holds true for sets in $\mathbb{R}^N$, for some $N \ge 2$, and more precisely, for all measures of the type

$$d\mu = x_N^k \exp\{c|x|^2\}\,dx$$

where $k \ge 0$ and $c \ge 0$.

We write $y = (x', x_N, x_{N+1})$ for points in $\mathbb{R}^{N+1}$, where $x' \in \mathbb{R}^{N-1}$, and $x_N, x_{N+1} \in \mathbb{R}$. Let a measure ν on

$$\mathbb{R}^{N+1}_+ := \{y = (x', x_N, x_{N+1}) \in \mathbb{R}^{N+1} : x_{N+1} > 0\}$$

be given by

$$d\nu = x_{N+1}^k \exp\{c(|x'|^2 + x_N^2 + x_{N+1}^2)\}\,dy.$$

We define two measures ν_1 and ν_2 by

$$d\nu_1 = \exp\{c|x'|^2\}\,dx', \quad \text{and}$$
$$d\nu_2 = x_{N+1}^k \exp\{c(x_N^2 + x_{N+1}^2)\}\,dx_N dx_{N+1},$$

and note that $d\nu = d\nu_1 d\nu_2$.

Let M be a subset of $\mathbb{R}^{N+1}_+$ having finite and positive ν-measure.

We define 2-dimensional slices

$$M(x') := \{(x_N, x_{N+1}) : (x', x_N, x_{N+1}) \in M\}, \quad (x' \in \mathbb{R}^{N-1}).$$

Let $M' := \{x' \in \mathbb{R}^{N-1} : 0 < \nu_2(M(x'))\}$, and note that $\nu_2(M(x')) < +\infty$ for a.e. $x' \in M'$. For all those x', let $H(x')$ be the half disc in $\mathbb{R}^2_+$ centered at $(0,0)$ with $\nu_2(M(x')) = \nu_2(H(x'))$. (For convenience, we put $H(x') = \emptyset$ for all $x' \in M'$ with $\nu_2(M(x')) = +\infty$.) By Theorem 3.2, we have

$$P_{\nu_2}(H(x')) \le P_{\nu_2}(M(x')) \quad \text{for a.e. } x' \in M'. \tag{3.54}$$

Let

$$H := \{y = (x', x_N, x_{N+1}) : (x_N, x_{N+1}) \in H(x'),\ x' \in M'\}.$$

The product structure of the measure ν tells us that

(i) $\nu(M) = \nu(H)$, and

(ii) the isoperimetric property for slices, (3.54), carries over to M, that is,

$$P_\nu(H) \le P_\nu(M), \tag{3.55}$$

(see for instance Theorem 4.2 of [38]).

Note again, the slice $H(x') = \{(x_N, x_{N+1}) : (x', x_N, x_{N+1}) \in H\}$ is a half disc $\{(r\cos\theta, r\sin\theta) : 0 < r < R(x'),\ \theta \in (0, \pi)\}$, with $0 < R(x') < +\infty$, $(x' \in M')$. Set

$$K := \{(x', r) : 0 < r < R(x'),\ x' \in M'\},$$

and introduce a measure α on $\mathbb{R}^N_+$ by

$$d\alpha := a_k r^{k+1} \exp\{c(|x'|^2 + r^2)\}\, dx' dr,$$

where

$$a_k := \int_0^\pi \sin^k\theta\, d\theta = B\left(\frac{k+1}{2}, \frac{1}{2}\right).$$

An elementary calculation then shows that

$$\nu(H) = \alpha(K),$$

and

$$P_\nu(H) = P_\alpha(K).$$

Let B_R denote the open ball in $\mathbb{R}^N$ centered at the origin, with radius R, and choose $R > 0$ such that

$$\alpha(B_R \cap \mathbb{R}^N_+) = \alpha(K).$$

By the induction assumption it follows that

$$P_\alpha(B_R \cap \mathbb{R}^N_+) \le P_\alpha(K). \tag{3.56}$$

Finally, let $M^\star$ be the half ball in $\mathbb{R}^{N+1}_+$ centered at the origin, with radius R,

$$M^\star := \{y = (x', x_N, x_{N+1}) : |x'|^2 + x_N^2 + x_{N+1}^2 < R^2,\ x_{N+1} > 0\}.$$

Then

$$\nu(M^\star) = \nu(M)$$

and

$$P_\nu(M^\star) = P_\alpha(B_R \cap \mathbb{R}^N_+).$$

Together with (3.55) and (3.56) we find

$$P_\nu(M^\star) \le P_\nu(M),$$

that is, the isoperimetric property holds for $N + 1$ in place of N dimensions. The theorem is proved. □

4. Application to a class of degenerate elliptic equations

4.1. Notation and preliminary results

First we introduce the notion of weighted rearrangement. For an exhaustive treatment of rearrangements we refer to [21,39–42].

Let the measure μ be given by (1.7), and let M be a measurable subset of $\mathbb{R}^N_+$. The distribution function of a Lebesgue measurable function $u : M \to \mathbb{R}$, with respect to $d\mu$, is the function m_μ defined by

$$m_\mu(t) = \mu\left(\{x \in M : |u(x)| > t\}\right), \quad \forall t \ge 0.$$

The decreasing rearrangement of u is the function u^* defined by

$$u^*(s) = \inf\left\{t \ge 0 : m_\mu(t) \le s\right\}, \quad \forall s \in (0, \mu(M)].$$

Let C_μ be the μ-measure of $B_1 \cap \mathbb{R}^N_+$, that is,

$$C_\mu = \frac{1}{2}(N-1)\omega_{N-1} B\left(\frac{k+1}{2}, \frac{N-1}{2}\right),$$

and let a function $\psi(r)$ be defined by

$$\psi(r) = \int_0^r \exp\left(ct^2\right) t^{N+k-1} dt.$$

Let $M^\star$ be defined as in Theorem 1.1, that is,

$$M^\star = B_{r^\star} \cap \mathbb{R}^N_+, \tag{4.1}$$

where

$$r^\star = \psi^{-1}\left(\frac{\mu(M)}{C_\mu}\right). \tag{4.2}$$

The rearrangement $u^\star$ of u, by its definition given in (1.10), is

$$u^\star(x) = u^*\left(C_\mu \psi\left(|x|\right)\right), \quad \forall x \in M^\star.$$

The isoperimetric inequality in Theorem 3.1 can be also stated as follows

$$P_\mu(M) \ge I_\mu(\mu(M)),$$

where $I_\mu(\tau)$ is the function such that $P_\mu(M^\star) = I_\mu(\mu(M^\star))$, or equivalently

$$I_\mu(\tau) = C_\mu \exp\left(c\left[\psi^{-1}\left(\frac{\tau}{C_\mu}\right)\right]^2\right)\left[\psi^{-1}\left(\frac{\tau}{C_\mu}\right)\right]^{N+k-1}. \tag{4.3}$$

The fact that half balls $B_R \cap \mathbb{R}^N_+$ are isoperimetric for the weighted measure μ imply a Polya-Szegö-type inequality (see [43], p. 125).

Theorem 4.1. *Let D be an open set with finite μ-measure, and let the space $V^2(D, d\mu)$ be given by Definition 2.1 Then we have for every function $u \in V^2(D, d\mu)$,*

$$\int_D |\nabla u|^2 d\mu \ge \int_{D^\star} |\nabla u^\star|^2 d\mu. \tag{4.4}$$

Since rearrangements preserve the L^p norms, we have that the Rayleigh-Ritz quotient decreases under rearrangement, i.e.

$$\frac{\int_D |\nabla u|^2 d\mu}{\int_D u^2 d\mu} \ge \frac{\int_{D^\star} |\nabla u^\star|^2 d\mu}{\int_{D^\star} (u^\star)^2\, d\mu}, \quad \forall u \in V^2(D, d\mu).$$

The following Poincarè type inequality states the continuous embedding of $V^2(D, d\mu)$ in $L^2(D, d\mu)$. It is a consequence of some one-dimensional inequalities (see [32], Theorem 2, p. 40).

Corollary 4.1. *Let D be an open subset of $\mathbb{R}^N_+$. Then there exists a constant C, such that for every $u \in V^2(D, d\mu)$,*

$$\int_D u^2 d\mu \le C \int_D |\nabla u|^2\, d\mu.$$

4.2. Comparison result

Now we are in a position to obtain sharp estimates for the solution to problem (1.8). By a weak solution to such a problem we mean a function u belonging to $V^2(D, d\mu)$ such that

$$\int_D A(x)\nabla u \nabla \chi\, d\mu = \int_D f\chi\, d\mu, \tag{4.5}$$

for every $\chi \in C^1(\bar{D})$ such that $\chi = 0$ on the set $\partial D \setminus \{x_N = 0\}$.

Proof of Theorem 1.2. Note first that the existence of a unique solution to problems (1.8) and (1.11) is ensured by the Lax–Milgram Theorem. Arguing as in [15] (see for instance [12], p. 363), we get

$$1 \le \left\{\left[I_\mu\left(m_u(t)\right)\right]^{-2} \int_0^{m_u(t)} f^*(\sigma) d\sigma\right\}\left(-m_u'(t)\right) \tag{4.6}$$

and

$$u^*(s) \le \int_s^{\mu(D)} \left(I_\mu^{-2}(l) \int_0^l f^*(\sigma) d\sigma\right) dl. \tag{4.7}$$

Using (4.3) in (4.7) we obtain

$$\begin{aligned}
u^\star(x) &\le \frac{1}{{C_\mu}^2}\int_{C_\mu\psi(|x|)}^{\mu(D)}\left\{\exp\left(-2c\left[\psi^{-1}\left(\frac{l}{C_\mu}\right)\right]^2\right)\left(\psi^{-1}\left(\frac{l}{C_\mu}\right)\right)^{-2N-2k+2}\int_0^l f^*(\sigma)d\sigma\right\}dl\\
&= \frac{1}{C_\mu}\int_{|x|}^{r^\star}\exp\left(-c\eta^2\right)\eta^{-N-k+1}\left(\int_0^{C_\mu\psi(\eta)}f^*(\sigma)d\sigma\right)d\eta \quad \left(\eta := \psi^{-1}\left(\frac{l}{C_\mu}\right)\right)\\
&= \int_{|x|}^{r^\star}\exp\left(-c\eta^2\right)\eta^{-N-k+1}\left(\int_0^{\eta}f^*(C_\mu\psi(\xi))\xi^{N+k-1}\exp\left(c\xi^2\right)d\xi\right)d\eta \quad \left(\sigma := C_\mu\psi(\xi)\right)\\
&= \int_{|x|}^{r^\star}\exp\left(-c\eta^2\right)\eta^{-N-k+1}\left(\int_0^{\eta}f^\star(\xi)\xi^{N+k-1}\exp\left(c\xi^2\right)d\xi\right)d\eta\\
&= w(x).
\end{aligned}$$

Now let us show (1.13). Arguing as in [12], p. 363–364 (see also [15]), we derive

$$\begin{aligned}
-\frac{d}{dt}\int_{|u|>t}|\nabla u|^q\,d\mu &\le \left(\int_0^{m_u(t)}f^*(s)ds\right)^{q/2}\left(-m_u'(t)\right)^{1-q/2}\\
&\le (I(m_u(t)))^{-q}\left(\int_0^{m_u(t)}f^*(s)ds\right)^q\left(-m_u'(t)\right).
\end{aligned}$$

Integrating the last inequality between 0 and $+\infty$, we get

$$\begin{aligned}
\int_D|\nabla u|^q\,d\mu &= \int_0^{+\infty}\left[-\frac{d}{dt}\int_{|u|>t}|\nabla u|^q\,d\mu\right]dt\\
&\le \int_0^{+\infty}\left(I_\mu(m_u(t))\right)^{-q}\left(\int_0^{m_u(t)}f^*(\sigma)d\sigma\right)^q\left(-m_u'(t)\right)dt\\
&\le \int_0^{\mu(D)}\left(I_\mu(s)\right)^{-q}\left(\int_0^s f^*(\sigma)d\sigma\right)^q ds.
\end{aligned}$$

Now a straightforward calculation yields

$$\begin{aligned}
\int_D|\nabla u|^q\,d\mu &\le C_\mu\int_0^{\mu(D)}\exp\left(-qc\left[\psi^{-1}\left(\frac{s}{C_\mu}\right)\right]^2\right)\left[\psi^{-1}\left(\frac{s}{C_\mu}\right)\right]^{-q(N+k-1)}\left(\int_0^s f^*(\sigma)d\sigma\right)^q ds\\
&= C_\mu^2\int_0^{R^\star}\exp\left(-qc\eta^2\right)\eta^{-q(N+k-1)}\left(\int_0^{C_\mu\psi(\eta)}f^*(\sigma)d\sigma\right)^q\exp\left(c\eta^2\right)\eta^{N+k-1}d\eta\\
&= C_\mu^2\int_0^{R^\star}\exp\left((1-q)c\eta^2\right)\eta^{(1-q)(N+k-1)}\left(\int_0^{C_\mu\psi(\eta)}f^*(\sigma)d\sigma\right)^q d\eta\\
&= C_\mu^2\int_0^{R^\star}\left(\int_0^{\eta}f^*(C_\mu\psi(\rho))C_\mu\exp\left(c\rho^2\right)\rho^{N+k-1}d\rho\right)^q\exp\left((1-q)c\eta^2\right)\eta^{(1-q)(N+k-1)}d\eta\\
&= C_\mu^{2+q}\int_0^{R^\star}\left(\int_0^{\eta}f^\star(\rho)\exp\left(c\rho^2\right)\rho^{N+k-1}d\rho\right)^q\exp\left((1-q)c\eta^2\right)\eta^{(1-q)(N+k-1)}d\eta\\
&= \int_D|\nabla w|^q\,d\mu. \quad \square
\end{aligned}$$

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