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Positivity and radial symmetry of solutions to some variational problems in $\mathbb{R}^N$ ☆

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Dedicated to the memory of Herbert Beckert (1920–2004)

Abstract

We minimize functionals

$$J(v_1, \dots, v_n) \equiv \int\limits_{\mathbb{R}^N} (1/p) \sum_{i=1}^{n} |\nabla v_i|^p - F\big(|x|, v_1, \dots, v_n\big)$$

in $(W^{1,p}(\mathbb{R}^N))^n$, subject to integral constraints

$$\int\limits_{\mathbb{R}^N} G_{ij}(v_i) = c_{ij} \quad (j = 1, \dots, k_i,\ i = 1, \dots, n).$$

We prove, under fairly weak conditions on the functions F, G_{ij}, that smooth minimizers are radially symmetric and do not change sign. We also show generalizations of this result to other variational problems associated to degenerate elliptic systems. Our proofs are based on rearrangement arguments and the strong maximum principle.

Keywords: Variational problem; Ground state solution; p-Laplacian; Entire solution; Radial symmetry of the solution; Elliptic equation; Rearrangement

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1. Introduction

Consider the following variational problem:

$$J(v) := \int_{\mathbb{R}^N} \left(\frac{|\nabla v|^p}{p} - F(v) \right) dx \to \mathrm{Inf}!,$$

$$\text{w.r.t. } v \in W^{1,p}(\mathbb{R}^N), \int_{\mathbb{R}^N} G(v)\, dx = 1, \tag{1}$$

where F and G are smooth functions satisfying suitable growth conditions. Problems of this type give rise to semilinear elliptic problems

$$-\Delta_p u \equiv -\nabla\left(|\nabla u|^{p-2}\nabla u\right) = h(u) \quad \text{in } \mathbb{R}^N, \quad \text{and} \quad \lim_{|x|\to\infty} u(x) = 0 \tag{2}$$

(with $h = F' + \alpha G'$, for some $\alpha \in \mathbb{R}$, in our case), and they have been extensively studied in the literature (see, e.g., [3,11,12,15,19,20,27,29] and references cited therein).

If both F and G are *even* functions, then there are no sign-changing minimizers of (1). Moreover, if u is a minimizer, then $(-u)$ is a minimizer, too, and if $u \geqslant 0$, then for a.e. $t > 0$, the level set $\{u > t\}$ is a ball and $|\nabla u| = \mathrm{const}$ a.e. on $\{u = t\}$. The proof of this well-known result is based on a rearrangement argument, which we recall below.

Let u a minimizer of (1). Since F and G are even, $|u|$ is a minimizer, too. Let then U denote the symmetrically decreasing rearrangement of $|u|$ (= Schwarz symmetrization; for the definition see [14]). Notice that U is radially symmetric and radially nonincreasing. Then $U \in W^{1,p}(\mathbb{R}^N)$, $\int G(u) = \int G(|u|) = \int G(U)$, $\int F(u) = \int F(|u|) = \int F(U)$, and $\int |\nabla u|^p = \int |\nabla |u||^p \geqslant \int |\nabla U|^p$, by the very properties of the rearrangement. Hence U is a minimizer, too. The above argument also shows that

$$\int_{\mathbb{R}^N} \left|\nabla|u|\right|^p dx = \int_{\mathbb{R}^N} |\nabla U|^p\, dx.$$

Due to a result of Brothers and Ziemer [8] it then follows that for a.e. $t > 0$, the level set $\{|u| > t\}$ is a ball and $|\nabla|u|| = \mathrm{const}$ a.e. on $\{|u| = t\}$. In particular, this implies that the set $\{|u| > 0\}$ is either a ball, a halfspace or the whole $\mathbb{R}^N$. Hence u cannot change sign, and the assertion follows. We emphasize that the above argument fails if F or G are not even functions.

There is a vast literature on symmetry results for *nonnegative* solutions of (2)—the so-called *ground states*—including situations where the nonlinearity h depends on $|x|$, and also for some related cooperative elliptic systems. In most cases, the proofs are either based on the well-known moving plane method (see, e.g., [9,10,13,16,17,26]) or on a rearrangement device called continuous Steiner symmetrization (see [4,5]). However, both methods are not applicable if the solution of (2) *changes sign*.

O. Lopes has shown that in the Laplacian case, $p = 2$, minimizers of (1) are radially symmetric provided that $F, G \in C^2(\mathbb{R})$ (see [22]). His proof is based on a nice combination of reflexion arguments with the principle of unique continuation. He also proved an analogous result for some related elliptic system (see Remark 4 below), and he showed other symmetry results for variational problems in domains with radial symmetry (see [21]).

We emphasize however that it was left as an open question in [22], whether the minimizing solution (which is radially symmetric!) might change sign.

In this paper, we show that smooth minimizers of (1) are radially symmetric and *do not change sign*, provided that the nonlinearity in (2) satisfies suitable growth conditions near its zero points (see Theorem 2). In particular, if $p \in (1,2]$, and if $F, G \in C^{1,p-1}(\mathbb{R})$, then any smooth solution of problem (1) which satisfies (2) has this property (compare Corollary 2(3) and Remark 2(2b)). We also show similar results for vector-valued problems with many integral constraints. Our symmetry proof (see Section 2) is divided in two steps.

Using a reflexion device which is called *two-point rearrangement*, we first show that the sets $\{u > t\}$ for $t > 0$ (respectively $\{u < t\}$ for $t < 0$), are balls, and that $|\nabla u|$ is constant on each level set $\{u = t\}$ ($t \in \mathbb{R}$) (see Theorem 1). Then an application of the strong maximum principle leads to the full symmetry result (see Theorem 2 and Corollary 2).

Finally we give an example of a sign-changing minimizer for a variational problem associated to the p-Laplacian, with two integral constraints (see Section 3).

2. Main results

We first introduce some notation. Let $N \in \mathbb{N}$. For any points $x, y \in \mathbb{R}^N$, let (x, y) denote Euclidean scalar product and $|x|$ Euclidean norm, and let $\mathbb{R}_0^+ = [0, +\infty)$. If $R > 0$, $x_0 \in \mathbb{R}^N$, then let $B_R(x_0) := \{x\colon |x| < R\}$, and $B_R := B_R(0)$. If $t \in \mathbb{R}$ and u is a function defined on $\mathbb{R}^N$, then we will use the abbreviation $\{u > t\}$ for the superlevel set $\{x\colon u(x) > t\}$, and similarly for the sublevel set $\{u < t\}$ and for the level set $\{u = t\}$. If $N \geqslant 2$, and if $p \in (1, N)$, then let $p^* = Np/(N-p)$.

Throughout the paper, let $n \in \mathbb{N}$, $k_i \in \mathbb{N}$, $i = 1, \dots, n$, and $q \in [1, +\infty)$ fixed numbers, and let $M = M(s)$, $F = F(s, t_1, \dots, t_n)$, $G_{ij} = G_{ij}(t_i)$, fixed functions defined $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$, $j = 1, \dots, k_i$, $i = 1, \dots, n$, and satisfying

$$M \in C^1(\mathbb{R}_0^+), \quad M(0) = 0, \quad M \text{ nonnegative and strictly convex,} \tag{3}$$

$$\begin{aligned}
&F \text{ differentiable w.r.t. } t_1, \dots, t_n,\\
&F, F_{t_i} \text{ measurable in } s \text{ and continuous in } t_1, \dots, t_n,\\
&F(\cdot, t_1, \dots, t_n), F_{t_i}(\cdot, t_1, \dots, t_n) \in L^\infty(\mathbb{R}_0^+) \quad \forall (t_1, \dots, t_n) \in \mathbb{R}^n,\\
&G_{ij} \in C^1(\mathbb{R}), \quad M' =: m, \quad F_{t_i} =: f_i, \quad G'_{ij} =: g_{ij},\\
&F(s, 0, \dots, 0) = f_i(s, 0, \dots, 0) = G_{ij}(0) = g_{ij}(0) = 0,\\
&f_i(s, t_1, \dots, t_n) \text{ nonincreasing in } s \text{ and nondecreasing in } t_k\\
&\quad \text{for } k \in \{1, \dots, n\},\ k \neq i,\\
&\quad \forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n,\ j = 1, \dots, k_i,\ i = 1, \dots, n.
\end{aligned} \tag{4}$$

Setting

$$K = \Big\{ v \equiv (v_1, \dots, v_n) \in \big(L^q(\mathbb{R}^N)\big)^n\colon M\big(|\nabla v_i|\big), F\big(|\cdot|, v\big), G_{ij}(v_i) \in L^1(\mathbb{R}^N),$$

$$\left.\int_{\mathbb{R}^N} G_{ij}(v_i)\,dx = c_{ij},\ j=1,\dots,k_i,\ i=1,\dots,n\right\},\tag{5}$$

where $c_{ij} \in \mathbb{R}$, $j = 1,\dots,k_i$, $i = 1,\dots,n$, we then consider the following variational problem:

$$\text{(P)}\quad J(v) \equiv \int_{\mathbb{R}^N} \left(\sum_{i=1}^{n} M\big(|\nabla v_i|\big) - F\big(|x|, v\big)\right) dx \to \text{Inf}\,!, \quad v \in K.\tag{6}$$

We call u a *minimizer of* (P) if $u \in K$, and if $J(v) \geqslant J(u)\ \forall v \in K$.

Suppose that u is a minimizer and

$$u_i \in L^\infty(\mathbb{R}^N), \quad i = 1,\dots,n,\tag{7}$$

and assume moreover, that the following manifold condition holds:

If, for some constants $\beta_1,\dots,\beta_{k_i}$, the function $\sum_{j=1}^{k_i} \beta_j g_{ij}(t)$ vanishes on some interval $c < t < d$, then it vanishes everywhere on $\mathbb{R}$, $i = 1,\dots,n$. (8)

Then standard arguments in the calculus of variations show (see, e.g., [30]) that u is a distributional solution of the following system of elliptic PDE:

$$\begin{aligned}-\nabla\left(m\big(|\nabla u_i|\big)\frac{\nabla u_i}{|\nabla u_i|}\right) &= f_i\big(|x|, u\big) + \sum_{j=1}^{k_i} \alpha_{ij} g_{ij}(u_i)\\ &\equiv h_i\big(|x|, u\big) \quad \text{in } \mathbb{R}^N,\ i = 1,\dots,n,\end{aligned}\tag{9}$$

with $\alpha_{ij} \in \mathbb{R}$, $j = 1,\dots,k_i$, $i = 1,\dots,n$, as Lagrange multipliers. Notice that $h_i = h_i(s, t_1,\dots,t_n)$ ($(s, t_1,\dots,t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$), is nonincreasing in s and nondecreasing in t_j, $i, j = 1,\dots,n$, $j \neq i$, i.e., the system (9) is *cooperative*.

Henceforth we will only consider minimizers which satisfy

$$u_i \in C^1(\mathbb{R}^N), \quad i = 1,\dots,n.\tag{10}$$

Since $u_i \in L^q(\mathbb{R}^N)$ we then also have

$$u_i(x) \to 0 \quad \text{as } |x| \to \infty,\ i = 1,\dots,n.\tag{11}$$

Remark 1. We will not specify conditions which ensure existence and regularity of minimizers of (P). Notice however, that the assumptions (7) and (10) above are not too restrictive in many cases:

Consider, for instance, a problem for the p-Laplacian operator, that is $M(s) = s^p/p$, with $1 < p < N$, $q = p^*$, and suppose that the following growth conditions are fulfilled:

$$\begin{aligned}&\big|f_i(s, t_1,\dots,t_n)\big|,\ \big|g_{ij}(t_i)\big| \leqslant c\big(1 + |t|^r\big) \quad \forall (s, t_1,\dots,t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n,\\ &\quad j = 1,\dots,k_i,\ i = 1,\dots,n, \text{ for some } c > 0 \text{ and } r \in (1, p^* - 1).\end{aligned}$$

Then any minimizer of (P) is bounded (see, e.g., [29]). In view of Eq. (9) this implies that $u_i \in C^{1,\alpha}$, $\alpha \in (0,1)$, $i = 1,\dots,n$; see [28].

Moreover, there is a wide class of smooth functions M such that any bounded solution u_i of (9) is C^1, $i = 1,\dots,n$; see [18].

For the proofs of our symmetry results we need some preliminary settings. Given any $(N-1)$-hyperplane Σ, let H one of the two open halfspaces into which $\mathbb{R}^N$ is divided by Σ, and let σ_H denote reflexion in $\Sigma = \partial H$. For any $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we define its *two-point rearrangement w.r.t.* H by

$$w_H(x) := \begin{cases} \max\{w(x); w(\sigma_H x)\} & \text{if } x \in H, \\ \min\{w(x); w(\sigma_H x)\} & \text{if } x \in \mathbb{R}^N \setminus \bar{H}. \end{cases}$$

For convenience, we will sometimes also write $\sigma = \sigma_H$, and $T^H w = w_H$.

Notice that two-point rearrangements have been proved particularly useful in showing integral inequalities related to Steiner and cap symmetrizations (see [2,7]). Below we summarize some properties of this transformation.

Lemma 1 (see [7]).

(1) *If* $\psi \in C(\mathbb{R})$, $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, *and if* $\psi(w) \in L^1(\mathbb{R}^N)$, *then* $\psi(w_H) \in L^1(\mathbb{R}^N)$, *and*

$$\int_{\mathbb{R}^N} \psi(w_H)\,dx = \int_{\mathbb{R}^N} \psi(w)\,dx. \tag{12}$$

(2) *If* $w \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, *and if* $|\nabla w| \in L^p(\mathbb{R}^N)$ *for some* $p \in [1, +\infty]$, *then also* $|\nabla w_H| \in L^p(\mathbb{R}^N)$. *If, moreover,* $M(|\nabla w|) \in L^1(\mathbb{R}^N)$, *then*

$$\int_{\mathbb{R}^N} M\big(|\nabla w|\big)\,dx = \int_{\mathbb{R}^N} M\big(|\nabla w_H|\big)\,dx. \tag{13}$$

Notice that (12) and (13) follow from the fact that the two-point rearrangement T_H rearranges the values of v and of $|\nabla v|$ in the two corresponding points $x, \sigma_H x$ for a.e. $x \in \mathbb{R}^N$. We also mention that the restriction to nonnegative functions w in [7] is not essential for the proofs.

We will also need an integral inequality related to two-point rearrangement that has been proved in [6]. Here we add a careful analysis of the equality sign. For the convenience of the reader, the full proof is included in Appendix A.

Lemma 2. *Let* $v = (v_1, \ldots, v_n) \in (L^q(\mathbb{R}^N))^n$, *let* $F(|\cdot|, v) \in L^1(\mathbb{R}^N)$, *and let* $0 \in \bar{H}$. *Then*

$$\int_{\mathbb{R}^N} F\big(|x|, v_1, \ldots, v_n\big)\,dx \leqslant \int_{\mathbb{R}^N} F\big(|x|, T^H v_1, \ldots, T^H v_n\big)\,dx. \tag{14}$$

Furthermore, if, for some $i \in \{1, \ldots, n\}$, *the function* $(\partial F/\partial t_i)(r, t_1, \ldots, t_n)$ *is strictly decreasing in* r *and* $0 \in H$, *then the equality in* (14) *is achieved only if* $v_i = T^H v_i$. *Finally, if, for some numbers* $i, j \in \{1, \ldots, n\}$, $i \neq j$, *the function* $(\partial F/\partial t_i)(r, t_1, \ldots, t_n)$ *is strictly increasing in* t_j, *then the equality in* (14) *is achieved only if*

$$\big(v_i(x) - v_i(\sigma_H x)\big)\big(v_j(x) - v_j(\sigma_H x)\big) \geqslant 0 \quad \forall x \in H. \tag{15}$$

Now we are in a position to prove the first symmetry result.

Theorem 1. *Let* $u = (u_1, \ldots, u_n)$ *be a minimizer of* (P) *satisfying* (9)–(11). *Then for any* $i \in \{1, \ldots, n\}$ *the following hold*:

(1) $|\nabla u_i|$ *is constant on the set* $\{u_i = t\}$ $\forall t \in (\inf u_i, \sup u_i)$, *and, in particular* $|\nabla u_i| = 0$ *on the set* $\{u_i = 0\}$. *Furthermore, if* $\sup u_i > 0$, *then the superlevel sets* $\{u_i > t\}$ *are balls* $\forall t \in (0, \sup u_i)$, *and if* $\inf u_i < 0$, *then the sublevel sets* $\{u_i < t\}$ *are balls* $\forall t \in (\inf u_i, 0)$.

(2) *If the function* $f_i = f_i(r, t_1, \ldots, t_n)$ *is strictly decreasing in* r, *then* u_i *is radially symmetric and radially nonincreasing about* 0*—and, in particular, nonnegative—that is, there is a function* $v \in C^1((0, +\infty))$ *such that*

$$u_i(x) = v\big(|x|\big) \quad \text{and} \quad v'(r) \leqslant 0 \quad \text{for } 0 < |x| = r < +\infty. \tag{16}$$

(3) *If, for some* $j \in \{1, \ldots, n\}$, $j \neq i$, *the function* $f_i = f_i(r, t_1, \ldots, t_n)$ *is strictly increasing in* t_j, *then the functions* u_i *and* u_j *are equally ordered, that is*

$$\big(u_i(x) - u_i(y)\big)\big(u_j(x) - u_j(y)\big) \geqslant 0 \quad \forall x, y \in \mathbb{R}^N. \tag{17}$$

Proof. (1) First observe that if H is any halfspace with $0 \in \bar{H}$ then $(T^H u_1, \ldots, T^H u_n) \in K$ and $J(T^H u_1, \ldots, T^H u_n) \leqslant J(u)$, by Lemma 1. Hence, $(T^H u_1, \ldots, T^H u_n)$ is a minimizer of (P), too, and $J(T^H u_1, \ldots, T^H u_n) = J(u)$. In particular, this implies

$$\int_{\mathbb{R}^N} F\big(|x|, u\big)\,dx = \int_{\mathbb{R}^N} F\big(|x|, T^H u_1, \ldots, T^H u_n\big)\,dx \quad \forall \text{ halfspaces } H \text{ with } 0 \in \bar{H}. \tag{18}$$

Now fix $i \in \{1, \ldots, n\}$. Assume $\sup u_i > 0$, and let $t \in (0, \sup u_i)$. Setting $S_i(t) := \{u_i = t\}$ we choose $x, y \in S_i(t)$, $x \neq y$, and a halfspace $H \subset \mathbb{R}^N$ such that $y = \sigma_H x$, and such that $0 \in \bar{H}$. We claim that this implies

$$\nabla u_i(x) = \nabla\sigma_H\big(u_i(x)\big), \tag{19}$$

that is, the gradients of u_i at the points x and y are oppositely directed w.r.t. reflexion in ∂H. Indeed, if $\nabla u_i(x) \neq \nabla\sigma_H(u_i(x))$, then $\nabla T^H u_i$ is discontinuous across some C^1-hypersurface S, while $T^H u_i \in C^1(B_\varepsilon(x) \setminus S)$ ($\varepsilon > 0$, small). But this is impossible, since $(T^H u_1, \ldots, T^H u_n)$ is a minimizer, and hence it satisfies a system of the form (9)—with the Lagrangian multipliers α_{ij} possibly replaced by some other numbers α'_{ij}, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$.[1]

Repeating the above considerations for all points $z \in S_i(t)$ we find that

$$\nabla u_i(z) = \nabla u_i(x) - \frac{2(\nabla u_i(x), z - x)}{|z - x|^2}(z - x) \quad \forall z \in S_i(t),\ z \neq x. \tag{20}$$

This in particular means that $|\nabla u_i| = \text{const} =: c_i(t)$ on $S_i(t)$. Since $u_i \in C^1(\mathbb{R}^N)$, it follows that $|\nabla u_i| = c_i(t)$ on the set $\{u_i = t\}$, and since u_i decays at infinity, also $|\nabla u_i| = 0$ on the set $\{u_i(x) = 0\}$.

[1] With some more effort we may actually prove that $\alpha_{ij} = \alpha'_{ij}$, $j = 1, \ldots, k_i$, $i = 1, \ldots, n$, but that information is not needed here.

Assume $c_i(t) \neq 0$. Then $S_i(t)$ is locally a C^1-hypersurface and (20) shows that

$$\nu(z) = \nu(x) - \frac{2(\nu(x), z-x)}{|z-x|^2}(z-x) \quad \forall z \in S_i(t),\ z \neq x,$$

where $\nu(z)$ denotes the exterior normal to $\{u_i > t\}$ at z. By Lemma R (Appendix A) and since u_i decays at infinity this means that the superlevel set $\{u_i > t\}$ is a ball in this case.

Next let $t \in (0, \sup u_i)$ and $c_i(t) = 0$. Then we find a strictly decreasing sequence $\{t_k\}$ with $\lim_{k\to\infty} t_k = t$ and such that $c_i(t_k) \neq 0$, $k = 1, 2, \ldots$. Since $\{u_i > t\} = \bigcup_{k=1}^{\infty}\{u_i > t_k\}$, this means that the superlevel set $\{u_i > t\}$ is a ball in this case, too.

Similarly we can prove that any sublevel set $\{u_i < t\}$ is a ball and $|\nabla u_i| = \text{const}$ on $\{u_i = t\}$ if $\inf u_i < 0$ and if $t \in (\inf u_i, 0)$.

(2) Next assume that, for some $i \in \{1, \ldots, n\}$, the function $f_i = f_i(r, t_1, \ldots, t_n)$ is strictly decreasing in r. In view of (18), Lemma 2 tells us that $u_i = T^H u_i$ for any halfspace H with $0 \in H$. It is easy to see that this implies the symmetry property (16).

(3) Finally let, for some numbers $i, j \in \{1, \ldots, n\}$, $i \neq j$, the function $f_i = f_i(r, t_1, \ldots, t_n)$ be strictly increasing in t_j, and assume that (17) is not satisfied. Then there exist two density points $x, y \in \mathbb{R}^N$ of u_i and u_j such that $u_i(x) > u_i(y)$ and $u_j(x) < u_j(y)$. We choose a halfspace H with $0 \in \bar{H}$ such that $y = \sigma_H x$. Then, applying Lemma 2, we obtain that $\int_{\mathbb{R}^N} F(|x|, u)\,dx < \int_{\mathbb{R}^N} F(|x|, T^H u_1, \ldots, T^H u_n)\,dx$, a contradiction. The theorem is proved. □

From part (1) of Theorem 1 one easily obtains that minimizers of (P) are 'locally radially symmetric':

Corollary 1. *Let $u = (u_1, \ldots, u_n)$ be a minimizer of* (P) *satisfying* (9)–(11)*, and let A be a connected component of the set $\{x\colon \nabla u_i(x) \neq 0\}$, $i \in \{1, \ldots, n\}$. Then there are numbers $R_1, R_2 \in [0, +\infty]$ with $R_1 < R_2$, and a point $z \in \mathbb{R}^N$ such that $A = \{x\colon R_1 < |x - z| < R_2\}$, and u_i is radially symmetric in A, that is, there is a function $v \in C^1((R_1, R_2))$, such that $u_i(x) = v(|x - z|)$, $x \in A$. Moreover, u_i does not change sign in A. Finally, if $u_i > 0$* (*respectively $u_i < 0$*) *in A then $v'(r) < 0$* (*respectively $v'(r) > 0$*)*, $r \in (R_1, R_2)$.*

Proof. We use the notation of the previous proof. In view of part (1) of Theorem 1, we find two numbers $a, b \in \mathbb{R}$, $a < b$, such that $A = \{x\colon a < u_i(x) < b\}$, and for each $t \in (a, b)$ the level set $\{u_i = t\}$ is a ball with $c_i(t) > 0$. Moreover, since $c(0) = 0$, we have that either $a \geqslant 0$ or $b \leqslant 0$, that is, u_i does not change sign in A. Finally, since $u_i \in C^1(\mathbb{R}^N)$, an easy application of the method of steepest descent (see [1]) shows that the level sets $\{u_i = t\}$, $t \in (a, b)$, are concentric spheres. The last assertion of the corollary then follows immediately. □

Using the weak symmetry of minimizers of (P), we now intend to show their radial symmetry (and their positivity as well!) provided that the functions h_i, $i = 1, \ldots, n$, in (9) satisfy some growth conditions near their zero points. Here the key role in the proof is played by a general version of the strong maximum principle that has been shown recently by Pucci, Serrin and Zou (see [24,25]).

In the sequel, let

$$\mu(s) := sm(s) - M(s) \quad (s \geqslant 0).$$

Since M is strictly convex, μ is continuous and strictly increasing, and it has a continuous and strictly increasing inverse μ^{-1}. We denote by $\mathcal{A}_m$ the set of functions $\alpha \in C(\mathbb{R}_0^+)$ satisfying

$$\alpha(0) = 0, \qquad \alpha(t) > 0 \quad \text{for } t > 0, \quad \text{and} \quad \int_0^1 \frac{dt}{\mu^{-1}(\int_0^t \alpha(s)\,ds)} = +\infty.$$

Strong maximum principle (SMP) (see [24,25]). *Let Ω be a domain in $\mathbb{R}^N$, let $\partial\Omega$ be smooth in a neighborhood of $x_0 \in \partial\Omega$, and let $u \in C^1(\Omega \cup \{x_0\})$ satisfy $u(x_0) = 0$ and in the sense of distributions,*

$$-\nabla\left(m(|\nabla u|)\frac{\nabla u}{|\nabla u|}\right) \geqslant -\alpha(u), \quad u \geqslant 0 \text{ in } \Omega,$$

where $\alpha \in \mathcal{A}_m$. Then

$$\frac{\partial u}{\partial \nu}(x_0) \leqslant 0 \quad (\nu\text{: exterior normal}),$$

where the equality sign is attained only if $u \equiv 0$ in Ω.

Definition 1. Let $h \in C(\mathbb{R}_0^+ \times \mathbb{R}^n)$, $h = h(s, t_1, \ldots, t_n)$, and $i \in \{1, \ldots, n\}$.

(1) We say that h has property $H_+(i, \tau)$, respectively $H_-(i, \tau)$, if there holds: If $h(\sigma, \tau_1, \ldots, \tau_n) = 0$ for some $(\sigma, \tau_1, \ldots, \tau_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ with $\tau_i = \tau$, then there exists a function $\alpha \in \mathcal{A}_m$ such that

$$h(s, t_1, \ldots, t_n) \geqslant -\alpha(t_i - \tau) \quad \forall (s, t_1, \ldots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n \text{ with } t_i \in [\tau, +\infty), \text{ respectively} \tag{21}$$

$$h(s, t_1, \ldots, t_n) \leqslant \alpha(\tau - t_i) \quad \forall (s, t_1, \ldots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n \text{ with } t_i \in (-\infty, \tau]. \tag{22}$$

(2) We say that h has property $H(i, \tau)$, if there holds: If $h(\sigma, \tau_1, \ldots, \tau_n) = 0$ for some $(\sigma, \tau_1, \ldots, \tau_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ with $\tau_i = \tau$, then there exists a function $\alpha \in \mathcal{A}_m$ such that h satisfies either one of the conditions (21) or (22) of (1).

(3) We say that h is nice w.r.t. the variable t_i, if h has property $H(i, \tau)$ for any $\tau \in \mathbb{R}$.

Theorem 2. *Let $u = (u_1, \ldots, u_n)$ be a minimizer of* (P) *satisfying* (9)–(11). *Then for every $i \in \{1, \ldots, n\}$ the following hold*:

(1) *If the function h_i in* (9) *has property $H(i, \tau)$ for any $\tau \in \mathbb{R} \setminus \{0\}$ then u_i is radially symmetric in the sets $\{u_i > 0\}$ and $\{u_i < 0\}$. More precisely, if $\sup u_i > 0$, then*

$$\exists z_+ \in \mathbb{R}^N,\ R_{1+}, R_{2+} \in [0,+\infty],\ R_{1+} < R_{2+},\ v_+ \in C^1((R_{1+}, R_{2+})),$$
$$\text{such that } \{u_i > 0\} = \big\{x\colon |x - z_+| < R_{2+}\big\},\quad u_i(x) = v\big(|x - z_+|\big),$$
$$\text{and}\quad v'_+(r) < 0 \quad \text{for } R_{1+} < |x - z_+| = r < R_{2+}, \tag{23}$$

and if $\inf u_i < 0$*, then*

$$\exists z_- \in \mathbb{R}^N,\ R_{1-}, R_{2-} \in [0,+\infty],\ R_{1+} < R_{2-},\ v_- \in C^1((R_{1-}, R_{2-})),$$
$$\text{such that } \{u_i < 0\} = \big\{x\colon |x - z_-| < R_{2-}\big\},\quad u_i(x) = v_-\big(|x - z_-|\big),$$
$$\text{and}\quad v'_-(r) > 0 \quad \text{for } R_{1-} < |x - z_-| = r < R_{2-}. \tag{24}$$

(2) *If the function* h_i *in* (9) *has property* $H(i,0)$*, then* u_i *does not change sign.*
(3) *If the function* h_i *in* (9) *is nice w.r.t. the variable* t_i*, then* u_i *is radially symmetric and does not change sign. More precisely, if* $\sup u_i > 0$*, then* u_i *is nonnegative and satisfies condition* (23)*, and if* $\inf u_i < 0$*, then* u_i *is nonpositive and satisfies condition* (24).

Proof. (1) Let h_i have property $H(i,\tau)$ for any $\tau \in \mathbb{R} \setminus \{0\}$. Assume that $\sup u_i > 0$, and let $t \in (0, \sup u_i)$ with $c(t) > 0$. We define $t_2 := \inf\{s < t\colon c(\tau) > 0\ \forall \tau \in (s,t]\}$, and $t_1 := \sup\{s > t\colon c(\tau) > 0\ \forall \tau \in [t,s)\}$. By Theorem 1(1), we have that $t_2 \geqslant 0$, and by Corollary 1, we find a point $z_+ \in \mathbb{R}^N$, numbers $R_{1+}, R_{2+} \in [0,+\infty]$, $R_{1+} < R_{2+}$, and a function $v_+ \in C^1((R_{1+}, R_{2+}))$ such that $A := \{t_2 < u_i < t_1\} = \{x\colon R_{1+} < |x - z_+| < R_{2+}\}$, and $u_i(x) = v_+(|x - z_+|)$, and $v'_+(r) < 0$ for $R_{1+} < |x - z| = r < R_{2+}$. Notice that in view of the equation for u_i, $h_i = h_i(|x|, u(x))$ can be written in A as a function of $|x - z_+|$, too.

Now assume that $t_2 > 0$. Then $R_{2+} < +\infty$, and since $v'_+(R_{2+}) = 0$, the SMP tells us that we must have $h_i(|x|, u(x)) = \text{const} =: k \leqslant 0$ on $\partial B_{R_{2+}}(z_+)$. Assume that $k < 0$. Since h_i is continuous, we find some $\varepsilon > 0$ such that $h_i(|x|, u(x)) < 0$ in $B_{R_{2+}+\varepsilon}(z_+) \setminus B_{R_{2+}}(z_+)$. Since $u_i(x) \leqslant t_2$ in $B_{R_{2+}+\varepsilon}(z_+) \setminus B_{R_{2+}}(z_+)$, the SMP gives $v'_+(R_{2+}) < 0$, a contradiction. Thus we have $k = 0$, and since h_i has property $H(i, t_2)$, the SMP tells us again that we must have $v'_+(R_{2+}) < 0$, a contradiction! Hence $h_i(|x|, u(x)) = 0$ on $\partial B_{R_{2+}}(z_+)$ and $t_2 = 0$.

Next assume that $t_1 < \sup u_i$. Then $R_{1+} > 0$, $\{u_i \geqslant t_1\} = \overline{B_{R_{1+}}(z_+)}$ and $v'_+(R_{1+}) = 0$. Using the SMP analogously as above, we find that $h_i(|x|, u(x)) = 0$ on $\partial B_{R_{1+}}(z_+)$. Then the fact that h_i has the property $H(i, t_1)$, and that $v'_+(R_{1+}) = 0$, leads again to a contradiction. It follows that $t_1 = \sup u_i$. This proves (23). If $\inf u_i < 0$ then one shows analogously as above that (24) holds.

(2) Let h_i have property $H(i,0)$, and assume that both sets $\{u_i > 0\}$ and $\{u_i < 0\}$ are nonempty. It then follows from Theorem 1 that each of these sets is a ball or a halfspace. The SMP then tells us that $h_i(|x|, u(x)) = 0$ and $u_i(x) = 0$ on $\partial\{u_i > 0\} \cup \partial\{u_i < 0\}$. Assume that h_i satisfies condition (21). But then the SMP gives $|\nabla u_i| \neq 0$ on $\partial\{u_i > 0\}$, a contradiction. Similarly one obtains a contradiction if h_i satisfies condition (22).

(3) This property follows directly from parts (1) and (2). □

We can exclude the possibility of 'plateaus' at height 0, $\sup u_i$ and $\inf u_i$, by slightly sharpening the growth conditions for the function h_i in (9) at these levels.

Corollary 2. *Let* $u = (u_1, \dots, u_n)$ *be a minimizer of* (P) *satisfying* (9)–(11), *and let* $i \in \{1, \dots, n\}$. *Then the following hold*:

(1) *If* $\sup u_i > 0$ (*respectively* $\inf u_i < 0$), *and if the function* h_i *in* (9) *has property* $H_-(i, \sup u_i)$ (*respectively* $H_+(i, \inf u_i)$), *then the set* $\{x\colon u_i(x) = \sup u_i\}$ (*respectively* $\{x\colon u_i(x) = \inf u_i\}$) *is a single point.*

(2) *If the function* h_i *in* (9) *has both properties* $H_-(i, 0)$ *and* $H_+(i, 0)$, *then either* $u_i(x) > 0$, $u_i(x) < 0$ *or* $u_i(x) \equiv 0$ *on* $\mathbb{R}^N$.

(3) *In particular, if* h_i *has both properties* $H_-(i, \tau)$ *and* $H_+(i, \tau)$ *for any* $\tau \in \mathbb{R}$, *and if* u_i *is positive* (*respectively negative*), *then there exists a point* $z \in \mathbb{R}^N$ *and a function* $v \in C^1(\mathbb{R}_0^+)$ *such that* $u_i(x) = v(|x - z|)$, *and* $v'(r) < 0$ (*respectively* $v'(r) > 0$), *for* $0 < |x - z| = r < +\infty$.

Proof. (1) Let $\sup u_i > 0$, and assume that h_i has property $H_-(i, \sup u_i)$. By Theorem 1, we find a point $x_0 \in \mathbb{R}^N$, and $R \geqslant 0$ such that $\{x\colon u_i(x) = \sup u_i\} = \overline{B_R(x_0)}$. Assume $R > 0$. Then the maximum principle shows that $h_i(s, t_1, \dots, t_n) = 0$ whenever $t_i = \sup u_i$. Hence h_i satisfies condition (22). Applying the SMP to the set $\{x\colon u_i(x) < \sup u_i\} = \mathbb{R}^N \setminus \overline{B_R(x_0)}$ we then find that $|\nabla u_i| \neq 0$ on $\partial B_R(x_0)$, a contradiction. Hence $R = 0$.

Analogously one shows that if $\inf u_i < 0$, then the set $\{x\colon u_i(x) = \inf u_i\}$ is a single point.

(2) Assume that h_i satisfies both properties $H_-(i, 0)$ and $H_+(i, 0)$. Then u_i does not change sign by Theorem 2(2). Assume that $\sup u_i > 0$ and that $\{u_i = 0\} \neq \emptyset$. Then $\{u_i > 0\}$ is either a ball or a halfspace, which means that $h(|x|, u_i(x)) = 0$ on $\{u_i = 0\}$. Hence h_i satisfies condition (21) with $\tau = 0$. Applying the SMP then shows that we must have $|\nabla u_i(x)| \neq 0$ on $\partial\{u_i > 0\}$, which is impossible. Hence $u_i(x) > 0$ on $\mathbb{R}^N$.

Analogously one shows that if $\inf u_i < 0$ then $u_i(x) < 0$ on $\mathbb{R}^N$.

The assertion (3) then follows from Theorem 2(3), and from the assertions (1) and (2) above. □

Remark 2. Let us illustrate the conditions on the nonlinearities h_i required in Theorem 1(2) and Corollary 2.

(1) The function f_i is strictly decreasing in r if it is, for instance, of the form

$$f_i(r, t_1, \dots, t_n) = \sum_{k=1}^{m_i} a_{ik}(r) b_{ik}(t_1, \dots, t_n),$$

with continuous and positive functions b_{ik}, and with strictly decreasing functions a_{ik}, $k = 1, \dots, m_i$.

Furthermore, one obtains radial symmetry of minimizers u of (P) by combining several of the conditions given in Theorem 1. For instance, if the function $f_1 = f_1(r, t_1, \dots, t_n)$ is strictly decreasing in r, and if the functions $f_i = f_i(r, t_1, \dots, t_n)$ are strictly increasing in t_j, $j = 1, \dots, n$, $i = 2, \dots, n$, $j \neq i$, then any component u_i has property (16), $i = 1, \dots, n$.

(2) The function h_i has both properties $H_-(i, \tau)$ and $H_+(i, \tau)$ $\forall \tau \in \mathbb{R}$, $\tau \neq 0$, if there exist two numbers $a_i \leqslant 0$, $b_i \geqslant 0$ such that $h_i(s, t_1, \dots, t_m) \geqslant 0$ $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$

with $t_i \in [a_i, 0] \cup [b_i, +\infty)$ and $h_i(s, t_1, \dots, t_n) \leqslant 0$ $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$ with $t_i \in (-\infty, a_i] \cup [0, b_i]$. Notice that the above inequalities are difficult to check in general, since they also depend on the Lagrangian multipliers in (9). Furthermore, h_i has both properties $H_-(i, 0)$ and $H_+(i, 0)$ if f_i has these properties, and if there exists a function $\alpha \in \mathcal{A}_m$ such that

$$\left|g_{ij}(t)\right| \leqslant \alpha\left(|t|\right) \quad \forall t \in \mathbb{R},\ j = 1, \dots, k_i.$$

Finally, for certain differential operators, the required growth conditions for h_i in Corollary 2(3) are fulfilled if the functions f_i and g_{ij}, $j = 1, \dots, k_i$, $i = 1, \dots, n$, in (9) have additional smoothness properties:

(a) Assume that the differential operator is nondegenerate, that is $c_1 s \leqslant m(s) \leqslant c_2 s$ $\forall s \in \mathbb{R}_0^+$, for some numbers $0 < c_1 < c_2 < +\infty$. Then we may put $\alpha(s) = cs$ $(c > 0)$ in (21) and (22). Hence h_i has both properties $H_-(i, \tau)$ and $H_+(i, \tau)$ $\forall \tau \in \mathbb{R}$ if $f_i = f_i(r, t_1, \dots, t_n)$ and $g_{ij} = g_{ij}(t_i)$ satisfy a Lipschitz condition w.r.t. t_i, $j = 1, \dots, k_i$, uniformly w.r.t. the other variables, $i = 1, \dots, n$. Examples for such operators are $M(s) = s^2/2$ (Laplacian operator), and $M(s) = \sqrt{1 + s^2} - 1$ (minimal surface operator).

(b) Next assume that $M(s) = s^p/p$ (p-Laplace operator), and that $p \in (1, 2]$. Choosing $\alpha(s) = cs^{p-1}$ $(c > 0)$ in (21) and (22), we see that h_i has both properties $H_-(i, \tau)$ and $H_+(i, \tau)$ $\forall \tau \in \mathbb{R}$, provided that $f_i = f_i(s, t_1, \dots, t_n)$, $g_{ij} = g_{ij}(t_i)$, $j = 1, \dots, k_i$, satisfy a Hölder condition with exponent $(p - 1)$ w.r.t. t_i, uniformly $\forall (s, t_1, \dots, t_n) \in \mathbb{R}_0^+ \times \mathbb{R}^n$, $i = 1, \dots, n$.

(3) In the scalar case, that is if $n = 1$, we need not to restrict ourselves to bounded solutions. In fact, our results hold true—with obvious changes in the formulation and in the proofs—if condition (10) is replaced by

$$u \in C^1(U), \quad \text{where } U := \left\{x\colon \left|u(x)\right| < \infty\right\}, \text{ and } U \text{ is open.} \tag{25}$$

In particular, if u is a minimizer of (P), and if the singular set $\{u = +\infty\}$ (respectively $\{u = -\infty\}$) is nonempty, then it must be a single point.

Remark 3. *Other constraints*. Our results can be extended to situations where the admissible set K contains further or other constraints that are *invariant under two-point rearrangement*. Here are some examples:

(1) *Inequality constraints*. Assume that K includes constraints of the form $a_i \leqslant v_i \leqslant b_i$, with $a_i \leqslant 0 \leqslant b_i$, $i = 1, \dots, n$, and assume that minimizers u are smooth. Then $U_i := \{x\colon a_i < u_i < b_i\}$ is an open set, and u_i satisfies Eq. (9) on U_i, $i = 1, \dots, n$. Our symmetry results then follow analogously as above.

(2) *Volume constraints*. Let K include constraints of the form $v_i \geqslant 0$ and $|\{v_i > 0\}| = \lambda_i$, where $\lambda_i > 0$, and $|\cdot|$ denotes Lebesgue measure, and assume that u is a minimizer satisfying $u_i \geqslant 0$, $u_i \in C^1(\{u_i > 0\}) \cap C^{0,1}(\mathbb{R}^N)$, where $\{u_i > 0\}$ is an open set, $i = 1, \dots, n$. Then u_i satisfies Eq. (9) in $\{u_i > 0\}$, and our symmetry results remain valid in the set $\{u_i > 0\}$. This implies that $\{u_i > 0\}$ is some ball with measure λ_i, and u_i satisfies the Bernoulli-type boundary condition

$$\frac{\partial u_i}{\partial \nu} = \mu_i \quad \text{on } \partial\{u_i > 0\} \ (\nu\text{: interior normal}), \tag{26}$$

where $\mu_i \geqslant 0$, $i = 1, \dots, n$.

(3) *Artificial constraints*. Our method is also applicable when the integral constraints involve derivatives of the admissible functions. Below we restrict ourselves to an example in the scalar case, $n = 1$.

Let $\theta \in (0, 1)$, $c, d > 0$, $f \in C^1(\mathbb{R})$, $f(0) = 0$, and assume that $\limsup_{|t|\to\infty} |f(t)| > 0$,

$$0 \leqslant \frac{f(t)}{t} \leqslant \theta f'(t) \quad \text{and} \tag{27}$$

$$\begin{aligned} &\left|f(t)\right| \leqslant c\left(1 + |t|^r\right) \quad \forall t \in \mathbb{R},\ \text{with } r \in (1, 2^* - 1) \text{ when } N \geqslant 3, \\ &\quad \text{and } r > 1 \text{ and arbitrary, when } N \leqslant 2. \end{aligned} \tag{28}$$

Setting $F(t) := \int_0^t F(s)\,ds$, $t \in \mathbb{R}$, and

$$K_A := \left\{ v \in W^{1,2}(\mathbb{R}^N):\ v \not\equiv 0,\ \int_{\mathbb{R}^N} \left(|\nabla v|^2 + d|v|^2 - vf(v)\right) dx = 0 \right\},$$

we consider the following problem:

$$(\mathrm{P}_A) \quad J_A(v) := \int_{\mathbb{R}^{\mathbf{N}}} \left(\frac{|\nabla v|^2 + dv^2}{2} - F(v) \right) dx \to \mathrm{Inf}!, \quad v \in K_A.$$

The existence of minimizers of (P_A) was proved in [20, Theorem III.1]. Let u one of them. In view of (27), it is easy to check that $-\Delta u + du = f(u)$ in $\mathbb{R}^N$. Then assumption (28) ensures that $u \in C^1(\mathbb{R}^N)$ and that u decays at infinity. Furthermore, we have that $u_H \in K_A$ for any halfspace H, by Lemma 1. Finally, the function h defined by

$$h(t) := f(t) - dt, \quad t \in \mathbb{R},$$

has both properties $H_-(1, \tau)$ and $H_+(1, \tau)$ for any $\tau \in \mathbb{R}$ (corresponding to $M(s) = s^2/2$). Proceeding exactly as above we then deduce that u has the symmetry property of Corollary 2(3).

Remark 4. It is interesting to compare our results with the work of Orlando Lopes. In [22], he investigated the variational problem

$$J(v) := \int_{\mathbb{R}^N} \left(\sum_{i=1}^n \frac{|\nabla v_i|^2}{2} - F(v_1, \ldots, v_n) \right) dx \to \mathrm{Inf}!,$$

subject to

$$v = (v_1, \ldots, v_n) \in \left(W^{1,2}(\mathbb{R}^N)\right)^n \quad \text{and} \quad \int_{\mathbb{R}^N} G(v_1, \ldots, v_n)\,dx = 1,$$

where the functions F and G satisfy appropriate smoothness and growth conditions. Using reflexion arguments and the principle of unique continuation, he showed that any minimizer is radially symmetric. We emphasize that no cooperativity condition on the integrands F and G are required here! Therefore it is not possible to recover this result by using our two-point rearrangement technique. On the other hand, the proof in [22] uses trial functions which are not rearrangements of the solution. Therefore it seems difficult to generalize the result to problems with more than one integral constraint. We also mention that the result of [22] does not imply that the solutions are monotone in the radial variable—in particular, they might change sign.

3. A sign-changing minimizer

In [26], Serrin and Zou gave an example of a nonnegative weak C^1-solution of (2) with compact support and with a plateau at some positive level, which has the local symmetry property described in Corollary 1. In view of this example, it is natural to ask whether such a symmetry breaking can also happen for minimizers of problem (P). Below we will obtain an example of a sign-changing minimizer of (P) in the scalar case $n = 1$, which is essentially based on the construction in [26]:

Example. Let $1 < p < N$, $a \in (0, \min\{(1/2); (p-1)/p\})$, and $k := p - 1 - ap$. Notice that $k > 0$. Then define

$$w(x) \equiv v(|x|) := \begin{cases} [1 - |x|^{1/a}]^{1/a} & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geqslant 1. \end{cases}$$

Since $(1/a) > 2$, we have that $w \in C^{2,\alpha}(\mathbb{R}^N)$ $(\alpha > 0)$, and $v'(r) < 0$ for $r \in (0, 1)$. Furthermore, w satisfies weakly $-\Delta_p w = g(w)$ in $\mathbb{R}^N$, where g is given by

$$\begin{aligned} g(t) = &-(k+a)a^{-2p}\big[t(1-t^a)\big]^k\big[1-(a+1)t^a\big] \\ &+ (N-1)a^{2(1-p)}\big[t(1-t^a)\big]^{k+a}(1-t)^{-a}, \quad t \geqslant 0. \end{aligned}$$

Notice that $g \in C^\infty((0, 1))$, $g(0) = g(1) = 0$, and $g'(t) < 0$ for small $t > 0$. Furthermore, we have that $g \in C^k([0, 1])$ and $\lim_{t\to 0} g'(t) = -\infty$ if $p \in (1, 2]$, and $g \in C^{1,\alpha}([0, 1])$ for some $\alpha > 0$, if $p > 2$, and if a is small enough. We extend g onto $\mathbb{R}$ by setting $g(t) = 0$ for $t \in \mathbb{R} \setminus [0, 1]$. Setting $G(t) := \int_0^t g(s)\,ds$, $t \in \mathbb{R}$, we find that $G(1) > 0$, and

$$\int_0^1 \frac{dt}{|G(t)|^{1/p}} < \infty. \tag{29}$$

Using a well-known integral identity which is due to Pohozaev, Pucci and Serrin (see [23]), we have that

$$\int_{\mathbb{R}^N} G(w)\,dx = \frac{1}{p^*} \int_{\mathbb{R}^N} |\nabla w|^p\,dx =: c_0 > 0.$$

Setting

$$K_0 = \left\{ v \in L^{p^*}(\mathbb{R}^N)\colon\ \nabla v \in \big(L^p(\mathbb{R}^N)\big)^N,\ \int_{\mathbb{R}^N} G(v)\,dx = c_0 \right\}$$

and

$$K_1 = \left\{ v \in K_0 \colon \int\limits_{\mathbb{R}^N} G(-v)\,dx = c_0 \right\},$$

we then consider the following two variational problems:

$$(\mathrm{P}_k) \qquad \int\limits_{\mathbb{R}^N} |\nabla v|^p\,dx \to \mathrm{Inf}!, \quad v \in K_k,\ k = 0, 1.$$

Problem (P_0) has a nonnegative radially symmetric and radially nonincreasing minimizer $u_0 \in C^1(\mathbb{R}^N)$, $-\Delta_p u_0 = g(u_0)$ in $\mathbb{R}^N$, and $u_0(x) \to 0$ as $|x| \to \infty$ (see [11, Theorem 1]). By the maximum principle, this implies $0 \leqslant u_0 \leqslant 1$. Moreover, since G satisfies (29), u_0 must have compact support (see [25, Theorem 2]). Setting $J(v) := \int_{\mathbb{R}^N} |\nabla v|^p\,dx$, and

$$u_1(x) := u_0(x) - u_0(x - x_0), \quad x \in \mathbb{R}^N,$$

where $x_0 \in \mathbb{R}^N$, $|x_0| > 2\,\mathrm{diam}(\mathrm{supp}\,w)$, we have that $u_1 \in K_1$, and $J(u_1) = 2J(u_0)$. On the other hand, setting $v_+ := \max\{0; v\}$, $v_- := \max\{0, -v\}$ for $v \in K_1$, we have that $v_-, v_+ \in K_0$, and

$$J(v) = J(v_-) + J(v_+) \geqslant 2\inf\bigl\{J(h)\colon\ h \in K_0\bigr\} = 2J(u_0) \quad \forall v \in K_1.$$

Hence u_1 is a minimizer of problem (P_1). Notice that

$$-\Delta_p u_1 = g(u_1) - g(-u_1) \equiv h(u_1) \quad \text{in } \mathbb{R}^N,$$

and in accordance with Theorem 2(2), h does not have property $H(1,0)$.

We conclude our work with some

Open problems.

(1) Is there an alternative proof of Theorem 1 which does not rely on the smoothness of the minimizer?
(2) Let $n = 1$. Given any number $t \neq 0$, can one construct a minimizer $u \in C^1(\mathbb{R}^N)$ of problem (P) with symmetry breaking at level $u = t$?
(3) Prove (or disprove) that *local* minimizers of problem (P) satisfy the symmetry property (1) of Theorem 1.

Appendix A. Technical results

Proof of Lemma 2. We first show the following technical

Lemma A.1. *Let $r_+ \geqslant r_- \geqslant 0$, $a_i, b_i, c_i^+, c_i^- \in \mathbb{R}$ with $c_i^+ = \max\{a_i; b_i\}$, $c_i^- = \min\{a_i; b_i\}$, $i = 1, \dots, n$. Then*

$$\begin{aligned} &F(r_-, a_1, \dots, a_n) + F(r_+, b_1, \dots, b_n) \\ &\quad \leqslant F\bigl(r_-, c_1^+, \dots, c_n^+\bigr) + F\bigl(r_+, c_1^-, \dots, c_n^-\bigr). \end{aligned} \tag{A.1}$$

Furthermore, if, for some $i \in \{1,\dots,n\}$, the function $(\partial F/\partial t_i)(r,t_1,\dots,t_n)$ is strictly decreasing in r and if $r_+ > r_-$, then the equality in (A.1) *is achieved only if $a_i = c_i^+$. Finally, if for some numbers $i, j \in \{1,\dots,n\}$, $i \neq j$, the function $(\partial F/\partial t_i)(r,t_1,\dots,t_n)$ is strictly increasing in t_j, then equality in* (A.1) *is achieved only if there holds*

$$(a_i - b_i)(a_j - b_j) \geqslant 0. \tag{A.2}$$

Proof. By regrouping the variables $t_1,\dots,t_n$, if necessary, we may assume w.l.o.g. that there is some $k \in \{1,\dots,n\}$ such that $a_i = c_i^-$ for $1 \leqslant i \leqslant k$, and if $k < n$, then also $a_i = c_i^+$ for $i > k$. Introducing the vectors $v' = (c_1^-,\dots,c_k^-)$, $v'' = (c_{k+1}^-,\dots,c_n^-)$, $h' = (h_1,\dots,h_k)$, $h'' = (h_{k+1},\dots,h_n)$, where $h_i := c_i^+ - c_i^-$, $i = 1,\dots,n$, (A.1) reads as

$$\begin{aligned} I :=\ & F(r_-, v'+h', v''+h'') + F(r_+, v', v'') \\ & - F(r_-, v', v''+h'') - F(r_+, v'+h', v'') \geqslant 0. \end{aligned}$$

We have

$$I = \int_0^1 \sum_{i=1}^k h_i \big(F_{t_i}(r_-, v'+th', v''+h'') - F_{t_i}(r_+, v'+th', v'')\big)\, dt.$$

Now each summand in the integrand is nonnegative in view of the assumptions on F, proving the first assertion. Moreover, we have $I = 0$ only if $h_i = 0$ or

$$F_{t_i}(r_-, v'+th', v''+h'') = F_{t_i}(r_+, v'+th', v'') \quad \forall t \in (0,1),$$

for any $i \in \{1,\dots,k\}$. From this the assertions in the equality case of (A.1) follow easily. □

We now continue with the proof of Lemma 2. We have by Lemma A.1, and since $|x| \leqslant |\sigma x|$ $\forall x \in H$,

$$\begin{aligned} & F\big(|x|, v_1(x),\dots,v_n(x)\big) + F\big(|\sigma x|, v_1(\sigma x),\dots,v_n(\sigma x)\big) \\ & \quad \leqslant F\big(|x|, T^H v_1(x),\dots,T^H v_n(x)\big) + F\big(|\sigma x|, T^H v_1(\sigma x),\dots,T^H v_n(\sigma x)\big) \\ & \quad \forall x \in H. \end{aligned} \tag{A.3}$$

Integrating this inequality over H, the first assertion of Lemma 2 follows. Furthermore, if, for some $i \in \{1,\dots,n\}$, the function $(\partial F/\partial t_i)(r,t_1,\dots,t_n)$ is strictly decreasing in r, and if $0 \in H$, then we have that $|x| < |\sigma x|$ $\forall x \in H$, and hence equality in (A.3) is achieved only if $v_i(x) = T^H v_i(x)$, by Lemma A.1.

Finally, if for some numbers $i, j \in \{1,\dots,n\}$, the function $(\partial F/\partial t_i)(r,t_1,\dots,t_n)$ is strictly increasing in t_j, then Lemma A.1 tells us that equality in (A.3) is achieved only if (15) holds. □

Lemma R (Reflexion lemma). *Let U be a nonempty open set in $\mathbb{R}^N$ with C^1-boundary S, and assume that*

$$\nu(y) = \nu(z) - \frac{2(\nu(z), y-z)}{|y-z|^2}(y-z) \quad \forall y, z \in S \text{ with } y \neq z, \tag{A.4}$$

where $\nu(x)$ denotes the exterior normal to U at x. Then U is either a halfspace, a ball or the exterior of a ball in $\mathbb{R}^N$.

Proof. Assume that U is not a halfspace. Then there exist two points $y_1, y_2 \in S$ such that $\nu(y_1) \neq \nu(y_2)$. Letting

$$z_0 := y_1 + \frac{|y_2 - y_1|^2}{2(\nu(y_1), y_2 - y_1)}\nu(y_1),$$

we may assume w.l.o.g. that $z_0 = 0$. It is then easy to see that $|y_1| = |y_2| = r$ for some $r > 0$, and either (i) $\nu(y_i) = y_i/r$ $(i = 1, 2)$, or (ii) $\nu(y_i) = -y_i/r$ $(i = 1, 2)$.

We claim that (i) implies that U is a ball. Clearly it is enough to show that

$$|x| = r \quad \forall x \in S,\ x \neq \pm y_i\ (i = 1, 2). \tag{A.5}$$

Setting $a_i := (r|x|)^{-1}(x, y_i)$, we have $|a_i| < 1$, and using (A.4) we find,

$$r\nu(x) = y_i - \frac{2r^2 - 2a_i|x|r}{r^2 - 2a_i|x|r + |x|^2}(y_i - x) \quad (i = 1, 2). \tag{A.6}$$

Multiplying (A.6) with $x/(r|x|)$, we have

$$\frac{(\nu(x), x)}{|x|} = \frac{-a_i r^2 - a_i|x|^2 + 2r|x|}{r^2 - 2a_i|x|r + |x|^2} \quad (i = 1, 2). \tag{A.7}$$

Introducing the function

$$f(t) := \frac{-tr^2 - t|x|^2 + 2r|x|}{r^2 - 2t|x|r + |x|^2}, \quad t \in (-1, 1),$$

we find that

$$f'(t) = \frac{-(r^2 - |x|^2)^2}{(r^2 - 2t|x|r + |x|^2)^2} < 0 \quad \forall t \in (-1, +1).$$

By (A.7), this means that we must have $a_1 = a_2$. Going back to (A.6) we finally calculate

$$y_2 - y_1 = \frac{2r^2 - 2a_1|x|r}{r^2 - 2a_1|x|r + |x|^2}(y_2 - y_1),$$

which implies that $r = |x|$. This shows (A.5), and the claim is proved.

Similarly one shows in case (ii) that U is the exterior of a ball in $\mathbb{R}^N$. □

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