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A Multiplicity Result for the $p-$Laplacian Involving a Parameter

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Abstract. We study existence and multiplicity of positive solutions for the following problem

$$\begin{cases} -\Delta_p u = \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases},$$

where λ is a positive parameter, Ω is a bounded and smooth domain in $\mathbb{R}^N$, $p \in (1,N)$, $f(x,t)$ behaves, for instance, like $o(|t|^{p-1})$ near 0 and $+\infty$, and satisfies some further properties. In particular, our assumptions allow us to consider both positive and sign changing nonlinearitites f, the latter describing logistic as well as reaction–diffusion processes.

By using sub– and supersolutions and variational arguments, we prove that there exists a positive constant $\overline{\lambda}$ such that the above problem has at least two positive solutions for $\lambda > \overline{\lambda}$, at least one positive solution for $\lambda = \overline{\lambda}$ and no solution for $\lambda < \overline{\lambda}$. An important rôle plays the fact that local minimizers of certain functionals in the C^1–topology are also minimizers in $W_0^{1,p}(\Omega)$. We give a short new proof of this known result.

1. Introduction and statement of the results

During the last two decades the $p-$Laplacian operator, Δ_p, has received growing attention. This is due to the fact that it arises in various applications. For instance, in Fluid Mechanics, the shear stress $\vec{\tau}$ and the velocity ∇u of certain fluids are related via an equation of the form $\vec{\tau}(x) = a(x)\nabla_p u(x)$, where $\nabla_p u = |\nabla u|^{p-2}\nabla u$. Here $p > 1$ is an arbitrary real number. The case $p = 2$ corresponds to a Newtonian fluid, and models of Non–Newtonian fluids are given by $p \neq 2$. The equations of motion then involve $\mathrm{div}(a\nabla_p u)$, which reduces to $a\Delta_p u = a\mathrm{div}\nabla_p u$, provided that a is a constant. Notice that the $p-$Laplacian also appears in the study of torsional

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creep (elastic case for $p = 2$, plastic case as $p \to \infty$, see [16]), of flow through porous media ($p = \frac{3}{2}$, see [29]), or of glacial sliding ($p \in (1, \frac{4}{3}]$, see [26]).

In this work, we will focus on the study the multiplicity of weak solutions of the problem

$$(P)_\lambda \qquad \begin{cases} -\Delta_p u = \lambda f(x,u), & u \geq 0 \quad \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{2,\beta}$–boundary ($\beta \in (0,1)$), and $p \in (1,N)$. Observe that if the nonlinearity $f(x,t)$ behaves like $o(|t|^{p-1})$ near zero and infinity, then via variational techniques it is not difficult to prove the existence of two positive solutions for λ large enough. Furthermore, under the same assumptions, a simple calculus involving the first eigenvalue of the $p-$Laplacian allows us to prove non–existence for λ small enough.

Our goal is to specify the range of multiplicity for every $\lambda \in (0,\infty)$. More precisely, we establish the existence of a positive constant $\overline{\lambda}$ such that the problem has at least two positive solutions for $\lambda > \overline{\lambda}$, at least one positive solution for $\lambda = \overline{\lambda}$ and no positive solution for $\lambda < \overline{\lambda}$. The proofs are based on variational arguments and the sub– and supersolutions technique. An important rôle plays the fact that local minimizers of certain functionals in the C^1–topology are also minimizers in $W_0^{1,p}(\Omega)$. In the appendix we will give a short new proof of this known result.

Our assumptions on the nonlinearity f will be the following:

(H_1) $f : \overline{\Omega} \times [0,+\infty) \longrightarrow \mathbb{R}$ is a measurable function and $f(x,\cdot)$ is continuous, uniformly for a.e. $x \in \Omega$, and
$$|f(x,t)| \leq C(1+t^r) \quad \forall (x,t) \in \Omega \times [0,+\infty), \tag{1.1}$$
for some numbers $C > 0$ and $r \in [0, p^*-1)$, where $p^* = Np/(N-p)$.

(H_2) There exists a continuous nondecreasing function $g : [0,+\infty) \to [0,+\infty)$ satisfying $g(0) = 0$, and such that the mapping $t \longmapsto f(x,t) + g(t)$ is nondecreasing.

(H_3) $\lim_{t\to 0+} f(x,t)t^{1-p} = 0$, uniformly for every $x \in \overline{\Omega}$.

(H_4) $\limsup_{t\to+\infty} f(x,t)t^{1-p} \leq 0$, uniformly for every $x \in \overline{\Omega}$.

(H_5) There holds either

(i) $f(x,t) > 0$ for every $(x,t) \in \Omega \times (0,+\infty)$; or

(ii) there exists $\delta_1 > 0$ and a ball $B_{\varepsilon_0}(x_0) \subset \Omega$, ($\varepsilon_0 > 0$, $x_0 \in \Omega$), such that $F(x,t) > 0$ on $B_{\varepsilon_0}(x_0) \times (0,\delta_1]$, where $F(x,t) := \int_0^t f(x,s)\,ds$, and there exists $q > p-1$ such that the mapping $t \longmapsto t^{-q} f(x,t)$ is *strictly* decreasing on $(0,+\infty)$ for a.e. $x \in \Omega$.

Let us comment on the hypotheses above.

A model for assumption (H_1) is, for instance, $f(x,u) = a(x)g(u)$, where $a \in L^\infty(\Omega)$ and $g \in C(\mathbb{R},\mathbb{R})$ is subcritical. (H_1) and (H_2) are standard in order to apply the sub- and super–solutions method (see [3], and Lemma 2.1 below).

The assumptions (H_3) and (H_4) ensure that there is a number $\bar{\lambda} > 0$ such that

Problem $(P)_\lambda$ has a positive solution for $\lambda \geq \bar{\lambda}$, and has no solution for $0 < \lambda < \bar{\lambda}$. Notice that (H_3) is a natural condition to obtain two positive solutions, since there are well known uniqueness results in cases that the limit in (H_3) is positive, see for instance [9], [15], [21]. For the existence of branches of positive solutions for asymptotically equi–diffusive problems, see Ambrosetti, García and Peral [2]. In particular, [2] deals with the equation $-\Delta_p u = \lambda f(u)\,,\ u \in W_0^{1,p}(\Omega)$ where f satisfies, for instance, $\lim_{t\to+\infty} f(x,t)t^{1-p} = c > 0$. We emphasize that this case is not considered here (compare with Hypothesis (H_4)).
Finally, we use assumption (H_5) in obtaining appropriate subsolutions of our problem $(P)_\lambda$ (see the proofs of the Lemmata 3.2 and 3.4 below). (H_5) also ensures the existence of a solution if λ is large enough (see the proof of Lemma 3.3).

Additionally, we suppose the following condition on the nonlinearity f which allows us to obtain a second solution of our Problem $(P)_\lambda$ for $\lambda \geq \bar{\lambda}$.

(H_6) There exist numbers $c_0 \geq 0$, $\delta_0 > 0$ such that the mapping $t \longmapsto f(x,t) + c_0 t^{p-1}$ is nondecreasing for $(x,t) \in \Omega \times (0, \delta_0]$.

Our main result is the following

Theorem 1.1. *Assume that f satisfies the conditions (H_1)–(H_6). Then there exists a positive constant $\overline{\lambda}$ such that Problem $(P)_\lambda$ has at least two positive solutions for $\lambda > \overline{\lambda}$, at least one positive solution for $\lambda = \overline{\lambda}$ and no positive solution for $\lambda < \overline{\lambda}$.*

Let us first give a few applications of our main result.
Let $p-1 < q,\ r > 0$, and $a_1,\ a_2 \in L^\infty(\Omega)$, with a_1 nonnegative and $\operatorname{ess\,inf}_\Omega a_2 > 0$. Then the conclusions of Theorem 1.1 hold in any one of the following cases:

1. $f_1(x,t) = a_1(x)t^q(1 - a_2(x)t^r)$.
   Notice in this case, in order to apply Theorem 1.1, one first needs to show the equivalence with an appropriate truncated problem, (see Section 4).
2. $f_2(x,t) = a_2(x)\dfrac{t^q}{1+t^r}$. where $q < p - 1 + r$.
3. $f_3(x,t) = a_2(x)\ln(1 + t^q)$.

Observe that particular cases of f_1 have been considered by many authors, since this type of nonlinearity models, for instance, reaction–diffusion processes or logistic problems in population dynamics, (see [25], [30]). For example, when $a_1(x) = a_2(x) = 1$, a multiplicity result was obtained by Takeuchi in [31] under the restriction $p > 2$. Later Dong and Cheng [11] proved the same result for all $p > 1$. We notice that the Laplacian case was studied by Rabinowitz by combining critical point theory with the Leray–Schauder degree [28]. For more information about type f_1, see [19], [22], [30], [32], [33] [10], [15] and the cited references therein.

We also observe that the functions f_1, f_2, f_3 above may be written as

$$f(x,t) = t^q h(x,u)$$

where the mapping $t \longmapsto h(x,t)$ is *strictly* decreasing on $(0, +\infty)$ for a.e. $x \in \Omega$. Existence results for nonlinearities of this type have been analyzed by Cañada et al [8]. However, in contrast to the present paper, the authors did not study the multiplicity of solutions.

On the other hand, it is interesting to compare our Problem $(P)_\lambda$ with a *dual* case, as it has been investigated in the classical paper of Ambrosetti, Brezis and Cerami [1]. The model equation with a concave-convex nonlinearity considered in [1] is $-\Delta u = \lambda u^q + u^r$, $u \in H_0^1(\Omega)$, where $0 < q < 1 < r \leq 2^*$. The authors proved that there exists $\tilde{\lambda} > 0$ such that the problem above has at least two positive solutions for $0 < \lambda < \tilde{\lambda}$, at least one positive solution for $\lambda = \tilde{\lambda}$ and no positive solution for $\lambda > \tilde{\lambda}$. Further results on this type of nonlinearities are given in [12, 13]. The analogous situation for the p-Laplacian has been studied by García and Peral [18]. Finally note that in Ambrosetti, García and Peral also established in [2] the existence of sign-changing solutions.
There are some recent papers that deal with the compactness of the branches of solutions of similar equations which are also relevant in the context of this paper: Cabré and Sanchon [5] considered nonnegative solutions of $-\Delta_p u = f(x, u)$. Assuming that $f(x, u)$ grows like $(1+u)^m$, where $0 < m < m^*$ and m^* is some critical value, and introducing the notion of semi-stability, they proved that certain minimizers of the associated energy functional are semi-stable and bounded.
Castorina *et al* [6] studied minimal solution branches (u_λ, λ) of the equation $-\Delta_p u = \lambda h(x) f(u)$, with $0 < u < 1$, and $0 < \lambda < \lambda^*$, where h is positive and Hölder continuous, f behaves like $(1-u)^{-m}$ near $u = 1$, and λ^* is some critical value. They showed that the mapping $\lambda \mapsto u_\lambda$ is non-decreasing, and composed by semistable solutions.

Our work is organized as follows:

In section 2 we give some definitions and basics facts which we will be used throughout the article. The proof of Theorem 1.1 is presented in Section 3. Then we extend our result to even more general nonlinearities in Section 4. Finally we present a new proof of a well–known result, Lemma 2.2, in an Appendix, section 5.

2. Preliminaries. Sub– and supersolutions

Let $\lambda_1(\Omega)$ denote the first eigenvalue of the Dirichlet p-Laplacian.
For convenience we extend the function $f(x, t)$ for negative values of t,

$$\widetilde{f}(x,t) = \begin{cases} f(x,t) & \text{if } (x,t) \in \overline{\Omega} \times [0, +\infty) \\ 0 & \text{if } (x,t) \in \overline{\Omega} \times (-\infty, 0) \end{cases},$$

and we will work with the problem

$$(\widetilde{P})_\lambda \quad \begin{cases} -\Delta_p u = \lambda \widetilde{f}(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

instead of $(P)_\lambda$.

Definition 2.1. *A function* $\underline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *is said to be a subsolution of* $(\widetilde{P})_\lambda$ *if*

$$\begin{cases} \int_\Omega |\nabla \underline{u}|^{p-2}\nabla \underline{u} \cdot \nabla\phi \, dx \le \lambda \int_\Omega \widetilde{f}(x,\underline{u})\phi \, dx \\ \text{for every } \ \phi \in W_0^{1,p}(\Omega) \ \text{ with } \phi \ge 0, \ \text{ and} \\ \underline{u} \le 0 \quad \text{on } \partial\Omega. \end{cases}$$

(A function $v \in W^{1,p}(\Omega)$ *is said to be less than or equal to* $w \in W^{1,p}(\Omega)$ *on* $\partial\Omega$ *when* $\max\{0; v-w\} \in W_0^{1,p}(\Omega)$*).*
Furthermore, a function $\overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *is said to be a supersolution of* $(P)_\lambda$ *if*

$$\begin{cases} \int_\Omega |\nabla \overline{u}|^{p-2}\nabla \overline{u} \cdot \nabla\phi \, dx \ge \lambda \int_\Omega \widetilde{f}(x,\overline{u})\phi \, dx \\ \text{for every } \ \phi \in W_0^{1,p}(\Omega) \ \text{ with } \ \phi \ge 0, \ \text{ and} \\ \overline{u} \ge 0 \quad \text{on } \partial\Omega. \end{cases}$$

Finally, a function $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ *which is both a sub– and a supersolution, is called a solution of problem* $(\widetilde{P})_\lambda$*.*

Remark 2.1. First notice that every solution of $(\widetilde{P})_\lambda$ is nonnegative. To see this, use $u_- := \max\{0; -u\}$ as a test function in (2.1). Integration by parts then leads to

$$0 \le \int_\Omega |\nabla u_-|^p \, dx = -\lambda \int_\Omega \widetilde{f}(x,u)u_- \, dx \le 0,$$

which implies that $u_- = 0$ a.e. on Ω. Thus, u is a solution of problem $(\widetilde{P})_\lambda$ iff u is also a solution of problem $(P)_\lambda$. By the Strong Maximum Principle we then obtain that every nontrivial solution of $(\widetilde{P})_\lambda$ is positive in Ω. Conditions (H_1) and (H_4) imply that there is a constant $C > 0$ which depends only on $\widetilde{f}$ and λ such that every solution u of problem $(\widetilde{P})_\lambda$ satisfies

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C. \tag{2.2}$$

Indeed, by condition (H_4) we have that there is a constant c_λ which depends only on λ such that

$$\lambda \widetilde{f}(x,t) \le c_\lambda + (1/2)\lambda_1(\Omega)t^{p-1} \qquad \forall\,(x,t) \in \Omega \times [0,+\infty)\,.$$

This inequality implies that all solutions of $(\widetilde{P})_\lambda$ are uniformly bounded in the $W_0^{1,p}(\Omega)-$norm. It is well-known that this implies that all solutions $(\widetilde{P})_\lambda$ are uniformly bounded in the $L^\infty-$norm. By using the regularity results of Guedda and Veron [20] and Lieberman [24] we then obtain the estimate (2.2).

Finally, condition (H_3) implies that if u is nontrivial, then it is positive in Ω and satisfies

$$0 > \frac{\partial u}{\partial \nu} \quad \text{on } \ \partial\Omega, \quad (\nu : \ \text{ exterior normal }), \tag{2.3}$$

see [34].

The following auxiliary result is well-known (see e.g. [3]) and will be basic in our approach.

Lemma 2.1. *Consider Problem* $(\widetilde{P})_\lambda$ *under the hypotheses* (H_1)*–*(H_2)*. Let* $\underline{u}, \overline{u} \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ *be, respectively, a subsolution and a supersolution of Problem* $(\widetilde{P})_\lambda$*, with* $\underline{u}(x) \le \overline{u}(x)$ *a.e. in* Ω*. Then there exists a minimal (and, respectively, a maximal) weak solution* u_* *(resp.* u^**) for Problem* $(\widetilde{P})_\lambda$ *in the "interval"*

$$[\underline{u}, \overline{u}] = \{u \in L^\infty(\Omega) : \underline{u}(x) \le u(x) \le \overline{u}(x) \text{ a.e. in } \Omega\}.$$

In particular, every weak solution $u \in [\underline{u}, \overline{u}]$ *of* $(\widetilde{P})_\lambda$ *also satisfies* $u_*(x) \le u(x) \le u^*(x)$ *for a.e.* $x \in \Omega$*.*

The following Lemma is crucial in showing multiplicity of solutions. It has been shown in the case $p = 2$ by Brezis and Nirenberg in [4], in the case $p > 2$ by Guo et al in [23], and in the general case by García Azorero et al in [17]. The idea consists in analyzing a penalized minimization problem. The proofs given in [17] and [23] are quite technical since the constraint involves the gradient of the admissible functions. We will present a short new proof in the Appendix. Notice that our constraint merely involves a certain L^q–norm.

Lemma 2.2. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be a Caratheodory function which satisfies*

$$|f(x,t)| \le C(1 + |t|^r) \qquad \forall\,(x,t) \in \Omega \times \mathbb{R}, \tag{2.4}$$

for some numbers $C > 0$ *and* $r \in [0, p^* - 1)$*, and assume that* $u \in W_0^{1,p}(\Omega)$ *is a weak solution of*

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{2.5}$$

Let I *denote the energy functional associated with* (2.5)*, that is*

$$I(v) := \int_\Omega \left(\frac{1}{p} |\nabla v|^p - F(x,v) \right) dx, \quad v \in W_0^{1,p}(\Omega). \tag{2.6}$$

Assume finally that u *is a local minimizer of* I *in* $C_0^1(\overline{\Omega})$*, that is, there exists a number* $\varepsilon > 0$ *such that* $I(v) \ge I(u)$ *for every* $v \in C_0^1(\overline{\Omega})$ *satisfying* $\|v - u\|_{C^1(\overline{\Omega})} < \varepsilon$*. Then* u *is also a local minimizer in* $W_0^{1,p}(\Omega)$*.*

3. Proof of the main result

From the hypotheses (H_1), (H_3) and (H_4) it follows that there is a number $\lambda_0 > 0$ such that

$$\lambda_1(\Omega) t^{p-1} \ge \lambda_0 \widetilde{f}(x,t) \quad \forall (x,t) \in \overline{\Omega} \times [0, +\infty). \tag{3.1}$$

where $\lambda_1(\Omega)$ denotes the first eigenvalue of the Dirichlet $p-$Laplacian.

Lemma 3.1. *Problem* $(\widetilde{P})_\lambda$ *does not have any positive solution for* $\lambda < \lambda_0$*, where* λ_0 *is given by* (3.1)*.*

Proof. Suppose that $(\widetilde{P})_\lambda$ admits a positive solution u_λ for some $\lambda < \lambda_0$. Using u_λ as a test function in $(\widetilde{P})_\lambda$ we then obtain

$$\int_\Omega |\nabla u_\lambda|^p\,dx = \lambda \int_\Omega \widetilde{f}(x,u_\lambda)u_\lambda\,dx < \lambda_1(\Omega)\int_\Omega u_\lambda^p\,dx,$$

which contradicts to the variational characterization of $\lambda_1(\Omega)$. □

Lemma 3.2. *Suppose that for some $\lambda' > 0$, Problem $(\widetilde{P})_{\lambda'}$ admits a positive solution $u_{\lambda'}$. Then for every $\lambda > \lambda'$, Problem $(\widetilde{P})_\lambda$ has at least one positive solution.*

Proof. We first construct a subsolution $\underline{u}$.
1) Assume f satisfies (H_5), (i). Since $u_{\lambda'}$ is a solution of $(\widetilde{P})_{\lambda'}$, we have that

$$-\Delta_p\, u_{\lambda'} = \lambda' \widetilde{f}(x,u_{\lambda'}) \le \lambda \widetilde{f}(x,u_{\lambda'}) \quad \text{in } \ \Omega,$$

that is $u_{\lambda'} =: \underline{u}$ is a subsolution of Problem $(\widetilde{P})_\lambda$.
2) Assume that f satisfies (H_5), (ii). Then there exists a number $\sigma \in (0,1)$ such that $\lambda' = \lambda\sigma^{q-p+1}$. Hence

$$-\Delta_p\,(\sigma u_{\lambda'}) = \sigma^{p-1}\lambda'\widetilde{f}(x,u_{\lambda'}) < \lambda \widetilde{f}(x,\sigma u_{\lambda'}) \quad \text{in } \ \Omega\ , \tag{3.2}$$

that is $\sigma u_{\lambda'} =: \underline{u}$ is a subsolution.
A supersolution is constructed as follows: Let e be the solution of the following problem,

$$\begin{cases} -\Delta_p\, e = 1 & \text{in } \ \Omega \\ e = 0 & \text{on } \ \partial\Omega\,, \end{cases} \tag{3.3}$$

and set $e_0 := \sup\{e(x) :\ x \in \Omega\}$. From (H_4), and since $e, \underline{u} \in C^1(\overline{\Omega})$ and $\partial e/\partial\nu < 0$ on $\partial\Omega$, we have for every $k > 0$ and large enough,

$$ke(x) > \underline{u}(x) \quad \text{in } \Omega,\quad \frac{\partial(ke)}{\partial\nu} < \frac{\partial\underline{u}}{\partial\nu} \quad \text{on } \partial\Omega, \tag{3.4}$$

$$\lambda\lambda_1(\Omega) < k^{(p-1)/2}\lambda_0 \quad \text{and} \quad \lambda(e_0)^{p-1}\widetilde{f}(x,t) < t^{p-1}, \tag{3.5}$$

for all $(x,t) \in \Omega \times (\sqrt{k}, +\infty)$, where λ_0 is the number in (3.1). Then the inequalities (3.1) and (3.5) imply

$$\lambda\widetilde{f}(x,kt) < k^{p-1} \quad \forall (x,t) \in \Omega \times [0,e_0]. \tag{3.6}$$

Hence

$$-\Delta_p(ke(x)) = k^{p-1} > \lambda\widetilde{f}(x,ke(x)) \quad \text{in } \ \Omega, \tag{3.7}$$

that is, $ke =: \overline{u}$ is a supersolution of problem $(\widetilde{P})_\lambda$. Finally, using Lemma 2.1 there exists a positive solution u of Problem $(\widetilde{P})_\lambda$ satisfying $\underline{u} \le u \le \overline{u}$. □

In the following we will work with the energy functional associated to problem $(\widetilde{P})_\lambda$, that is

$$I_\lambda(v) := \int_\Omega \left(\frac{1}{p}|\nabla v|^p - \lambda\widetilde{F}(x,v)\right)dx, \quad v \in W_0^{1,p}(\Omega), \tag{3.8}$$

where $\widetilde{F}(x,t) := \int_0^t \widetilde{f}(x,s)\,ds$, $((x,t) \in \Omega \times \mathbb{R})$.

Lemma 3.3. *Let* $\overline{\lambda} := \inf\{\lambda : (\widetilde{P})_\lambda$ *has a positive solution* $\}$. *Then* $0 < \overline{\lambda} < +\infty$, *and* $(\widetilde{P})_\lambda$ *has a positive solution for every* $\lambda > \overline{\lambda}$ *and no positive solution for* $0 < \lambda < \overline{\lambda}$.

Proof. By our assumptions, I_λ is differentiable, bounded from below, and coercive. Hence there is a global minimizer of I_λ on $W_0^{1,p}(\Omega)$ which is solution of $(\widetilde{P})_\lambda$. In view of condition (H_5), I_λ attains negative values if λ is large enough. Hence we have that $I_\lambda(u_\lambda) < 0$. Using the Lemmata 3.1 and 3.2 this implies that there is a number $\overline{\lambda} \in (0, +\infty)$ such that problem $(\widetilde{P})_\lambda$ has a solution if $\lambda > \overline{\lambda}$, and no solution if $\lambda < \overline{\lambda}$. □

Lemma 3.4. *Let* $\lambda > \overline{\lambda}$ *and suppose that Problem* $(\widetilde{P})_\lambda$ *has a unique positive solution* u_λ. *Then* u_λ *is a local minimizer of* I_λ *in* $C_0^1(\overline{\Omega})$.

Proof. The idea is to construct a sub– and a supersolution which are strictly separated from the solution u_λ. Let $\overline{\lambda} < \lambda' < \lambda$, and define $\underline{u}$ and $\overline{u}$ as in the proof of Lemma 3.2. Since $u_\lambda \in C^1(\overline{\Omega})$, we may add the requirement that the function $\overline{u} = ke$ in (3.7) satisfies

$$\overline{u} > u_\lambda \quad \text{in } \Omega, \text{ and}$$
$$\frac{\partial u_\lambda}{\partial \nu} > \frac{\partial \overline{u}}{\partial \nu} \quad \text{on } \partial\Omega.$$

For the construction of a subsolution $\underline{u}$ we split into two cases:
1) Assume (H_5), (i). Then $\underline{u} = u_{\lambda'} \leq u_\lambda$ in Ω. Set

$$\Omega_{-r} := \{x \in \Omega : \text{dist } (x, \partial\Omega) > r\}, \quad (r > 0),$$

and notice that Ω_{-r} is a C^2–domain for small enough r. Since $u_\lambda, u_{\lambda'} \in C_0^1(\overline{\Omega})$, there is a number $r > 0$ such that $|\nabla u_{\lambda'}|, |\nabla u_\lambda| > 0$ on $\overline{\Omega} \backslash \Omega_{-2r}$ and $u_\lambda(x), u_{\lambda'}(x) \in (0, \delta_0)$ on $\Omega \setminus \overline{\Omega_{-2r}}$. Together with assumption (H_6) this in particular implies

$$\begin{aligned} -\Delta_p u_{\lambda'} + \lambda c_0 (u_{\lambda'})^{p-1} &= \lambda' \widetilde{f}(x, u_{\lambda'}) + \lambda c_0 (u_{\lambda'})^{p-1} < \lambda \left(\widetilde{f}(x, u_{\lambda'}) + c_0 (u_{\lambda'})^{p-1} \right) \\ &\leq \lambda \left(\widetilde{f}(x, u_\lambda) + c_0 (u_\lambda)^{p-1} \right) \\ &= -\Delta_p u_\lambda + \lambda c_0 (u_\lambda)^{p-1} \quad \text{in } \Omega \setminus \overline{\Omega_{-2r}}, \end{aligned}$$

in the distributional sense. The Strong Comparison Principle - see for instance Proposition 5.1 of [14], or Proposition 3 of [7] - then tells us that

$$u_{\lambda'} < u_\lambda \quad \text{in } \Omega \setminus \overline{\Omega_{-2r}}, \text{ and} \tag{3.9}$$
$$0 > \frac{\partial u_{\lambda'}}{\partial \nu} > \frac{\partial u_\lambda}{\partial \nu} \quad \text{on } \partial\Omega. \tag{3.10}$$

Hence there is a number $\varepsilon_0 > 0$ such that $u_\lambda \geq \varepsilon_0 + \underline{u}$ on $\partial\Omega_{-r}$. This implies that if $0 < \varepsilon < \varepsilon_0$ then $(u_{\lambda'} - u_\lambda + \varepsilon)_+ \in W_0^{1,p}(\Omega_{-r})$. Hence

$$\begin{aligned} 0 &\leq \int_{\Omega_{-r}} \left(|\nabla u_{\lambda'}|^{p-2}\nabla u_{\lambda'} - |\nabla u_\lambda|^{p-2}\nabla u_\lambda\right) \cdot \nabla(u_{\lambda'} - u_\lambda + \varepsilon)_+ \, dx \\ &= \int_{\Omega_{-r}} \Big(\lambda' \widetilde{f}(x, u_{\lambda'}) - \lambda \widetilde{f}(x, u_\lambda)\Big) (\underline{u} - u_\lambda + \varepsilon)_+ \, dx, \end{aligned} \tag{3.11}$$

for these ε, where the inequality in (3.11) follows from

$$\left(|x|^{p-2}x - |y|^{p-2}y\right) \cdot (x - y) \geq 0 \quad \forall x, y \in \mathbb{R}^N. \tag{3.12}$$

Since $\lambda' < \lambda$, $u_{\lambda'} \leq u_\lambda$ and $\widetilde{f}(x, u_{\lambda'}(x)) > 0$ in $\overline{\Omega_{-r}}$, and since $\widetilde{f}$ is continuous in the second variable, there exists a number $\varepsilon_1 \in (0, \varepsilon_0)$ such that

$$\lambda' \widetilde{f}(x, u_{\lambda'}(x)) - \lambda \widetilde{f}(x, u_\lambda(x)) < 0 \quad \text{on} \quad \{x \in \Omega_{-r} : \, u_{\lambda'}(x) + \varepsilon_1 > u_\lambda(x)\}.$$

In view of (3.11) this implies that $\underline{u} + \varepsilon_1 \leq u_\lambda$ in Ω_{-r}.
2) Assume (H_5), (ii). Then $\underline{u} = \sigma u_{\lambda'}$ is a subsolution of $(\widetilde{P})_\lambda$, where $\lambda' = \lambda\sigma^{q-p+1}$, and $\sigma u_{\lambda'} \leq u_\lambda$ in Ω. Choosing $r > 0$ as in case **1)** and taking into account that $\sigma^q \widetilde{f}(x, u_{\lambda'}) < \widetilde{f}(x, \sigma u_{\lambda'})$ in Ω, an analogous calculus shows that $\sigma u_{\lambda'} < u_\lambda$ in Ω and that (3.10) holds with $u_{\lambda'}$ replaced by $\sigma u_{\lambda'}$.
Now setting

$$\mathcal{A} := \{v \in C_0^1(\overline{\Omega}) : \, \underline{u} \leq v \leq \overline{u}\},$$

in any of the above cases, we find that u_λ is an *interior* point of $\mathcal{A}$ with respect to the C^1-topology. It is well-known that this implies that u_λ is a local minimizer of I_λ in $C_0^1(\overline{\Omega})$ (see [17], proof of Theorem 5.2). For the convenience of the reader we repeat the argument below. Let

$$\overline{f}(x, s) = \begin{cases} \widetilde{f}(x, \underline{u}(x)) & \text{if } \; s < \underline{u}(x) \\ \widetilde{f}(x, s) & \text{if } \; \underline{u}(x) \leq s \leq \overline{u}(x) \\ \widetilde{f}(x, \overline{u}(x)) & \text{if } \; \overline{u}(x) < s \end{cases},$$

$\overline{F}(x, t) := \int_0^t \overline{f}(x, s)\, ds$, $((x, t) \in \Omega \times \mathbb{R})$, and define a functional $\overline{I}_\lambda$ analogously as I_λ with F replaced by $\overline{F}$. From our assumptions on the nonlinearity f and standard arguments it follows that $\overline{I}_\lambda$ has a global minimizer $u_0 \in W_0^{1,p}(\Omega)$. Clearly u_0 is a weak solution of

$$\begin{cases} -\Delta_p u_0 = \lambda \overline{f}(x, u_0) & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.13}$$

Moreover, well-known regularity results (see e.g. [20]) show that $u_0 \in C_0^1(\overline{\Omega})$. On the other hand, using $(\underline{u} - u_0)_+$ as a test function for (3.13) and $(\widetilde{P})_\lambda$, we obtain, using (3.12),

$$\begin{aligned} 0 &\le \int_{\{\underline{u}>u_0\}} (|\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla u_0|^{p-2}\nabla u_0)\cdot(\nabla \underline{u} - \nabla u_0)\,dx \\ &\le \lambda \int_{\{\underline{u}>u_0\}} (\widetilde{f}(x,\underline{u}) - \overline{f}(x,u_0))(\underline{u}-u_0) = 0, \end{aligned}$$

which implies that $u_0 \ge \underline{u}$. Analogously one shows that $u_0 \le \overline{u}$. Hence we have that $u_0 \in \mathcal{A}$, which implies $u_\lambda = u_0$, by our hypothesis. Since u_λ is an interior point of $\mathcal{A}$, there exists a number $\varepsilon > 0$ such that one has $u \in \mathcal{A}$ for every $u \in C_0^1(\overline{\Omega})$ satisfying $\|u_\lambda - u\|_{C^1(\overline{\Omega})} < \varepsilon$. Furthermore, we find that for every $u \in \mathcal{A}$,

$$\begin{aligned} I_\lambda(u) - \overline{I}_\lambda(u) &= -\lambda \int_\Omega \Big(\widetilde{F}(x,u) - \overline{F}(x,u)\Big)\,dx \\ &= -\lambda \int_\Omega \int_0^u \Big(\widetilde{f}(x,s) - \overline{f}(x,s)\Big)\,ds\,dx \end{aligned}$$

which is a constant independent of u. Hence u_λ is a local minimizer of I_λ in $C_0^1(\overline{\Omega})$. □

Proof of Theorem 1.1. We argue by contradiction, i.e., we suppose that there is a number $\lambda > \overline{\lambda}$ such that Problem $(\widetilde{P})_\lambda$ has a unique positive solution u_λ. Then the Lemmata 3.4 and 2.2 imply that the solution u_λ is a local minimizer of I_λ in $W_0^{1,p}(\Omega)$. Furthermore, another local minimizer of I_λ on $W_0^{1,p}(\Omega)$ is given by 0. Since the functional I_λ is coercive, it follows that I_λ satisfies the (PS)–condition. Then applying an extended version of the Mountain Pass Theorem due to Pucci and Serrin [27], we obtain the existence of a *third* critical point of I_λ, which contradicts our assumption.

Finally we claim that problem $(\widetilde{P})_{\overline{\lambda}}$ has a positive solution. By assumption (H_3) there is a number $t_0 > 0$ such that

$$(\overline{\lambda}+1)\widetilde{f}(x,t) \le \frac{\lambda_1(\Omega)}{2} t^{p-1} \quad \forall (x,t) \in \Omega \times [0,t_0] \tag{3.14}$$

Let $\{\lambda_n\}$ a strictly decreasing sequence with $\lim_{n\to\infty} \lambda_n = \overline{\lambda}$, $\lambda_n \in (\overline{\lambda}, \overline{\lambda}+1)$, and let u_{λ_n} be a positive solution of $(\widetilde{P})_{\lambda_n}$, $n = 1, 2, \ldots$. We will show that (3.14) implies that

$$\|u_{\lambda_n}\|_\infty > t_0 \qquad n = 1, 2, \ldots. \tag{3.15}$$

In fact, assume that $u_{\lambda_n} \le t_0$ in Ω. Then condition (3.14) gives

$$\int_\Omega |\nabla u_{\lambda_n}|^p\,dx = \lambda_n \int_\Omega \widetilde{f}(x,u_{\lambda_n}) u_{\lambda_n}\,dx \le \frac{\lambda_1(\Omega)}{2} \int_\Omega u_{\lambda_n}^p\,dx,$$

which contradicts the variational characterization of $\lambda_1(\Omega)$. Hence we obtain (3.15). As in Remark 2.1 we then have that the functions u_{λ_n} are equibounded in $C^{1,\alpha}(\overline{\Omega})$. Thus by Arzela-Ascoli's Theorem there is a subsequence of $\{u_{\lambda_n}\}$ which converges in $C_0^1(\overline{\Omega})$ to a function $u \in C_0^1(\overline{\Omega})$ which can be identified easily as a solution of problem $(\widetilde{P})_{\overline{\lambda}}$, and (3.15) shows that

$$\|u\|_\infty \geq t_0. \tag{3.16}$$

This proves the claim. □

4. Concluding Remarks

We conclude our work with some examples and comments of our main result.

Assume first that f satisfies (H_1)–(H_4), (H_5),(ii), (H_6), except condition (1.1), but instead suppose that there exists a number $t_0 > 0$, such that $f(x, t_0) \leq 0$ for every $x \in \Omega$. Then any solution of the modified problem,

$$(\widehat{P}_\lambda) \quad \begin{cases} -\Delta u = \lambda \widehat{f}(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}, \tag{4.1}$$

where $\widehat{f}$ is the truncated function

$$\widehat{f}(x,t) = \begin{cases} \widetilde{f}(x,t) & \text{if } t \leq t_0 \\ \widetilde{f}(x,t_0) & \text{if } t > t_0 \end{cases}$$

satisfies $u(x) \leq t_0$ and is also a solution of $(P)_\lambda$. Notice that the truncated function $\widehat{f}$ satisfies (1.1). This means that we obtain at least two solutions for large enough λ even in situations when $\widetilde{f}$ has supercritical growth at infinity. However, it might be a difficult question, whether or not there are solutions of the original problem which are not solutions of the truncated problem in such a case. We illustrate this be means of a simple example: Let

$$f(x,t) = a(x)t^q|1 - t|,$$

where $p-1 < q < p^* - 2$ and $a \in L^\infty(\Omega)$. Then, it is easy to see that Problem $(P)_\lambda$ has a Mountain Pass solution for any $\lambda > 0$. By using the truncation argument at $t_0 = 1$ we obtain the existence of a positive constant $\overline{\lambda}$ such that the Problem $(\widehat{P})_\lambda$ has two solutions for $\lambda > \overline{\lambda}$ and no solution for $0 < \lambda < \overline{\lambda}$. This means in particular that the Mountain Pass solution u_λ of the Problem $(P)_\lambda$ satisfies $\|u_\lambda\|_{L^\infty(\Omega)} > 1$ for $0 < \lambda < \overline{\lambda}$.

On the other hand, the nonexistence of solutions for the original problem for λ small enough can be established, if there exists a positive constant t_0 such that $f(x,t) \leq 0$ for all $t \geq t_0$. Indeed, applying the Strong Maximum Principle, it is easy to see that any solution of the original problem is bounded from above by t_0, which means that the original and the truncated problem are equivalent in such a case.

Finally we mention that we may also deal with nonlinearities which involve nonpositive perturbations:

In fact, let f and k be functions satisfying conditions (H_1)–(H_4), (H_6), with f positive and k nonpositive on $\Omega \times (0, \infty)$, and consider the problem

$$(K)_\lambda \begin{cases} -\Delta_p u = \lambda f(x,u) + k(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} . \tag{4.2}$$

By slightly modifying the proofs of Lemma 3.2 and Lemma 3.4 in case **1)**, it easy to see that the conclusions of Theorem 1.1 hold for the family of problems $(K)_\lambda$, $(\lambda > 0)$.

We emphasize that in the context of multiplicity, nonlinearities as in (4.2) have been studied only in the case that both f and k are homogeneous in the second variable, and more precisely, for $f(x,t) = a(x)t^q$, $k(x,t) = -b(x)t^r$, $(a, b \in L^\infty(\Omega)$, $a, b > 0$, $r > q > p - 1)$, (see [10]). However, a simple linear transformation shows that the families $(P)_\lambda$ and $(K)_\lambda$ are equivalent in such a case.

5. Appendix

Proof of Lemma 2.2. First observe that due to the growth conditions on f and the smoothness of Ω we have that $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, see [20]. Let $q \in (r, p^* - 1)$, where r is the number from (2.4),

$$K(w) := \frac{1}{q+1} \int_\Omega |w(x) - u(x)|^{q+1}\, dx, \quad (w \in W_0^{1,p}(\Omega)),$$

and

$$S_\varepsilon := \{v \in W_0^{1,p}(\Omega) : K(v) \le \varepsilon\}, \quad (\varepsilon > 0).$$

Assume that the conclusion of Lemma 2.2 is not true. Then there exists for every $\varepsilon \in (0,1]$ a function $v_\varepsilon \in S_\varepsilon$ such that $I(v_\varepsilon) < I(u)$. Moreover, we may assume w.l.o.g., that v_ε is a global minimizer of I in S_ε.
In the following, let C be a generic positive constant which may vary from line to line, and is independent of ε, and let $\langle \cdot, \cdot \rangle$ denote the duality product between $W_0^{1,p}(\Omega)$ and its dual space. We consider two cases.

1) Let $K(v_\varepsilon) < \varepsilon$. Then v_ε is also a local minimizer of I in $W_0^{1,p}(\Omega)$. Hence v_ε is a solution of (2.5), which implies that

$$\|v_\varepsilon\|_{C^{1,\alpha}(\overline{\Omega})} \le C, \tag{5.1}$$

where $\alpha \in (0,1)$, and the constant C depends only on Ω, $\|u\|_{L^\infty}$ and q.

2) Let $K(v_\varepsilon) = \varepsilon$. Then there exists a number $\mu_\varepsilon \in \mathbb{R}$ - a Lagrangean multiplier - such that $I'(v_\varepsilon) = \mu_\varepsilon K'(v_\varepsilon)$, i.e., v_ε is a weak $W_0^{1,p}$–solution of

$$\begin{cases} -\Delta_p v_\varepsilon = \gamma(\mu_\varepsilon, x, v_\varepsilon) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where γ is defined by

$$\gamma(s,x,t) := f(x,t) + s|t - u(x)|^{q-1}(t - u(x)), \quad ((s,x,t) \in \mathbb{R} \times \overline{\Omega} \times \mathbb{R}).$$

First assume that $\mu_\varepsilon > 0$. Then there exists $h \in W_0^{1,p}(\Omega)$ such that $\langle I'(v_\varepsilon), h\rangle < 0$ and $\langle K'(v_\varepsilon), h\rangle < 0$, which implies $\langle K'(v_\varepsilon), h\rangle = \mu_\varepsilon^{-1}\langle I'(v_\varepsilon), h\rangle < 0$. Using Taylor's theorem, this means that there exists a number $\tau_0(=\tau_0(\varepsilon)) > 0$ such that $K(v_\varepsilon + \tau h) < K(v_\varepsilon)$ and also $I(v_\varepsilon + \tau h) < I(v_\varepsilon)$ for every $\tau \in (0, \tau_0)$. The first of these inequalities implies that $(v_\varepsilon + \tau h) \in S_\varepsilon$ for these τ. Hence v_ε is not a global minimizer of I in S_ε, a contradiction. It follows that $\mu_\varepsilon \le 0$.
First suppose $\mu_\varepsilon \in [-1, 0]$. Since $u \in L^\infty(\Omega)$, we have in view of (2.4),

$$|\gamma(s, x, t)| \le C(1 + |t|^q) \quad \forall (s, x, t) \in [-1, 0] \times \Omega \times \mathbb{R}$$

where C is a positive constant depending only on $\|u\|_\infty$, q and Ω. Now, $\varepsilon \le 1$ implies that $\|v_\varepsilon\|_{L^{q+1}} \le C(\|u\|_\infty, \Omega, q)$. Hence, this inequality together with the equation (5.2) give us a $W_0^{1,p}(\Omega)-$estimate depending only on $\|u\|_\infty$, q and Ω. Thus v_ε is bounded in $L^\infty(\Omega)-$norm by a constant which does not depend on ε. Therefore v_ε is bounded in $C^{1,\alpha}(\Omega)-$norm by a constant which does not dependent on ε (see [24]). This implies (5.1) in this case.
Suppose finally that $\mu_\varepsilon \le -1$. Since there exists a number $M > 0$, which is independent on ε, such that

$$\begin{aligned} &\gamma(s, x, t) < 0 \qquad \forall (s, x, t) \in (-\infty, -1] \times \Omega \times (M, +\infty), \quad \text{and} \\ &\gamma(s, x, t) > 0 \qquad \forall (s, x, t) \in (-\infty, -1] \times \Omega \times (-\infty, -M), \end{aligned}$$

the Maximum Principle tells us that $|v_\varepsilon(x)| \le M$ in Ω. Using $(v_\varepsilon - u)|v_\varepsilon - u|^{\beta-1}$, with $\beta \ge 1$, as a test function in (2.5) and (5.2) we obtain,

$$\begin{aligned} 0 \quad &\le \quad \beta \int_\Omega (|\nabla v_\varepsilon|^{p-2}\nabla v_\varepsilon - |\nabla u|^{p-2}\nabla u) \cdot \nabla(v_\varepsilon - u)|v_\varepsilon - u|^{\beta-1}\,dx \\ &= \quad \int_\Omega (f(x, v_\varepsilon) - f(x, u))\,(v_\varepsilon - u)\,|v_\varepsilon - u|^{\beta-1}\,dx + \mu_\varepsilon \int_\Omega |v_\varepsilon - u|^{\beta+q}\,dx. \end{aligned}$$

In view of the bounds for u and v_ε and Hölder's inequality this implies

$$-\mu_\varepsilon \|v_\varepsilon - u\|^q_{L^{\beta+q}(\Omega)} \le C|\Omega|^{q/(q+\beta)},$$

where C does not depend on β and ε. Passing to the limit $\beta \to +\infty$ this leads to

$$-\mu_\varepsilon \|v_\varepsilon - u\|^q_{L^\infty(\Omega)} \le C.$$

Hence we have that

$$|\gamma(\mu_\varepsilon, x, v_\varepsilon(x))| \le C \quad \text{in} \ \ \Omega,$$

from which we again obtain (5.1).
Thus we have shown that the functions v_ε, $(0 < \varepsilon \le 1)$, are equibounded in $C^{1,\alpha}(\overline{\Omega})$. Using Ascoli–Arzela's Theorem we find a sequence $\varepsilon_n \searrow 0$ such that

$$v_{\varepsilon_n} \to u \quad \text{in} \ \ C^1(\overline{\Omega}).$$

Since $I(v_{\varepsilon_n}) < I(u)$, we then have that u is not a local minimizer of I in $C_0^1(\overline{\Omega})$, a contradiction. $\square$

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