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A General Rearrangement Inequality à la Hardy–Littlewood

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Let $F = F(v_1, \dots, v_m)$ smooth on $(\mathbf{R}_0^+)^m$ with $F_{v_i v_j} \geq 0$ for $i \neq j$. Furthermore, let $u_1, \dots, u_m$ nonnegative and bounded functions on $\mathbf{R}^n$ with compact support. We prove the inequality $\int_{\mathbf{R}^n} F(u_1, \dots, u_m)\, dx \leq \int_{\mathbf{R}^n} F(u_1^*, \dots, u_m^*)\, dx$, where * denotes symmetric decreasing rearrangement.

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1 INTRODUCTION

Let $u : X \to \mathbf{R}_0^+$ a measurable function defined on a measure space (X, μ). We define its *distribution function* $\mu_u : \mathbf{R}_0^+ \to \mathbf{R}_0^+ \cup \{\infty\}$ by

$$\mu_u(t) = \mu(\{x \in \mathbf{R}^n : u(x) > t\}), \quad t \geq 0.$$

Two functions u and v are said to be *rearrangements of each other* if

$$\mu_u(t) = \mu_v(t) \quad \forall t \geq 0.$$

By $\mathcal{F}_+(X)$ we denote the collection of measurable functions $u : X \to \mathbf{R}_0^+$ with

$$\mu_u(t) < +\infty \quad \forall t > 0.$$

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Next let $\mathcal{M}$ the family of μ-measurable sets in X, and let $T : \mathcal{M} \to \mathcal{M}$ a set transformation satisfying

$$M, N \in \mathcal{M}, \quad M \subset N \Rightarrow TM \subset TN \text{ (monotonicity)}, \tag{1}$$

$$M \in \mathcal{M} \Rightarrow \mu(M) = \mu(TM) \text{ (equimeasurability)}. \tag{2}$$

Defining a function Tu by

$$Tu(x) := \sup\{t \geq 0 : \ x \in T(\{u > t\})\}, \quad x \in X, \tag{3}$$

we see that u and Tu are rearrangements of each other. We will also call any mapping $T : \mathcal{F}_+(X) \to \mathcal{F}_+(X)$ given by (1)–(3), a rearrangement. Now the following inequality is basic: if $u, v \in \mathcal{F}_+(X)$ and if T is a rearrangement then

$$\int_X uv \, \mathrm{d}\mu(x) \leq \int_X TuTv \, \mathrm{d}\mu(x). \tag{4}$$

Equation (4) is attributed to Hardy and Littlewood (see [10],[11]).

An extension of this inequality was proved by Crowe *et al.* [7]. Let $F : \mathbf{R}_0^+ \times \mathbf{R}_0^+ \to \mathbf{R}$ be continuous, and suppose that

$$F(0,0) = 0 \tag{5}$$

and

$$F(u+s, v+t) - F(u+s, v) - F(u, v+t) + F(u, v) \geq 0 \quad \forall u, v, s, t \in \mathbf{R}_0^+. \tag{6}$$

Then we have for $u, v \in \mathcal{F}_+(X)$,

$$\int_X F(u, v) \, \mathrm{d}\mu(x) \leq \int_X F(Tu, Tv) \, \mathrm{d}\mu(x). \tag{7}$$

Note that if $F \in C^2$ then (6) is equivalent to

$$\frac{\partial^2 F}{\partial u \partial v} \geq 0,$$

and if $\mu(X) < \infty$ then the condition (5) is superfluous.

Equation (7) contains some important special cases: if $F(u, v) = uv$ then we recover (4). Furthermore, if $F(u, v) = -f(|u - v|)$, where f is convex with $f(0) = 0$ then F is continuous and satisfies (6). In particular, for $f(t) = t^p$, $p \in [1, +\infty)$, we obtain the *nonexpansivity in* $L^p(X, \mu)$, i.e. for nonnegative functions $u, v \in L^p(X, \mu)$ there holds

$$\|Tu - Tv\|_p \leq \|u - v\|_p. \tag{8}$$

(Here and in the following $\|\cdot\|_p$ denotes the usual norm in $L^p(X, \mu)$.)

Our aim is to generalize (7) to more than two functions: suppose that $F : (\mathbf{R}_0^+)^m \to \mathbf{R}$ is continuous and satisfies

$$F(0, \dots, 0) = 0 \tag{9}$$

and

$$\begin{aligned} &F(v_1, \dots, v_i + s, \dots, v_j + t, \dots, v_m) - F(v_1, \dots, v_i + s, \dots, v_j, \dots, v_m) \\ &\quad - F(v_1, \dots, v_i, \dots, v_j + t, \dots, v_m) + F(v_1, \dots, v_i, \dots, v_j, \dots, v_m) \geq 0 \\ &\qquad \forall s, t, v_1, \dots, v_m \in \mathbf{R}_0^+ \text{ and } \forall i, j \in \{1, \dots, m\},\ i \neq j. \end{aligned} \tag{10}$$

Note that if $F \in C^2$ then (10) is equivalent to

$$\frac{\partial^2 F}{\partial v_i \partial v_j} \geq 0 \quad \forall i, j \in \{1, \dots, m\},\ i \neq j.$$

We ask for conditions on (X, μ) and T such that the following inequality might hold ($u_1, \dots, u_m \in \mathcal{F}_+(X)$):

$$\int_X F(u_1, \dots, u_m)\, \mathrm{d}\mu(x) \leq \int_X F(Tu_1, \dots, Tu_m)\, \mathrm{d}\mu(x). \tag{11}$$

We first point out that (11) was proved by Lorentz [12] for decreasing rearrangement of functions given on a bounded interval on $\mathbf{R}$. It seemed difficult to use his idea of proof for other rearrangements.

We will prove (11) in the special cases that T is the *symmetric decreasing rearrangement* either in the Euclidean space $\mathbf{R}^n$, on the sphere $\mathcal{S}^n$ or in the hyperbolic space $\mathbf{H}^n$ (for definitions see Section 2). The idea consists in the following: first (11) is shown for an elementary T – the so-called *two-point rearrangement*. Then the result follows by approximation through a sequence of two-point rearrangements. Note that this method of proof turned out to be very fruitful in showing integral inequalities for symmetrizations (see e.g. [1–4]).

Using (11) for symmetric decreasing rearrangement, it is also easy to obtain analogous inequalities for symmetrizations depending essentially on more than one variable. Unfortunately, regardless of the simplicity of the method, it cannot be applied to more general situations. In particular, we do not know whether (11) holds for arbitrary rearrangements T and measure spaces (X, μ).

Finally we show using (11) that vector valued solutions of certain variational problems are radially symmetric.

2 SYMMETRIZATIONS AND TWO-POINT REARRANGEMENT

From now on, let X denote either $\mathbf{R}^n$, the unit n-dimensional sphere $\mathcal{S}^n = \{x \in \mathbf{R}^{n+1}\colon |x| = 1\}$, or the n-dimensional hyperbolic space $\mathbf{H}^n$. We equip X with the corresponding distance function $d(x, y)$ and measure μ. Thus, for $X = \mathbf{R}^n$, $d(x, y)$ is the usual Euclidean distance $|x - y|$ and μ the Lebesgue measure. For $X = \mathcal{S}^n$, $d(x, y)$ is the great circle distance on the sphere and μ is the Lebesgue surface measure. We take as a model for $\mathbf{H}^n$ the ball $\{x \in \mathbf{R}^n\colon |x| < 1\}$ equipped with Riemannian line element $dh = 2(1 - |x|^2)^{-1}|\mathrm{d}x|$. Then d and μ ae the Riemannian distance function and the volume measure, respectively, associated with dh.

If $u \in C(X)$ we denote by ω_u the *modulus of continuity* of u, which is defined by

$$\omega_u(t) := \sup\{|u(x) - u(y)|\colon d(x, y) \leq t\}, \quad t \geq 0.$$

We fix an origin e in X, and we denote by $B(R)$ the ball $\{x \in X\colon d(x, e) < R\}$. For sets $M \subset \mathcal{M}$ with $\mu(M) > 0$ let M^* the ball $B(R)$, $R \in (0, +\infty]$, with $\mu(B(R)) = \mu(M)$, with the exception that, if $X = \mathcal{S}^n = M$, then $M^* = \mathcal{S}^n$. For $u \in \mathcal{F}_+(X)$ we define the *symmetric decreasing rearrangement* u^* by

$$u^*(x) = \sup\{t \geq 0\colon x \in \{u(x) > t\}^*\}. \tag{12}$$

Then u^* is 'radially symmetric and radially decreasing', that is we have $u^*(x)=u^*(y)$ if $d(e,x)=d(e,y)$ and $u^*(x)\geq u^*(y)$ if $d(e,x)\leq d(e,y)$. Furthermore, it is well-known (see [1]) tht if $u\in C(X)$ then $\omega_u(t)\geq\omega_{u^*}(t)$ $\forall t\geq 0$. Note that u^* goes by various other names, such as *Schwarz symmetrization* when $X=\mathbf{R}^n$, *spherical symmetrization* when $X=\mathcal{S}^n$, and *hyperbolic symmetrization* when $X=\mathbf{H}^n$.

Next we define a very simple rearrangement.

Let $\mathcal{H}(\mathbf{R}^n)$ the collection of all $(n-1)$-dimensional hyperplanes of $\mathbf{R}^n$, $\mathcal{H}(\mathcal{S}^n)$ the collection of all intersections of $\mathcal{S}^n$ with n-hyperplanes through the origin in $\mathbf{R}^{n+1}$, and $\mathcal{H}(\mathbf{H}^n)$ the collection of all images under the group of hyperbolic motions of the hyperbolic $(n-1)$-plane $\{x\in\mathbf{R}^n\colon |x|<1, x_n=0\}$. For $H\in\mathcal{H}(X)$ let $\sigma=\sigma_H: X\to X$ denote reflection in H, and let H^+ and H^- are the two components of $X\backslash H$, such that $e\in H\cup H^+$.

For $u\in\mathcal{F}_+(X)$ we define the *two-point rearrangement* of u (w.r.t. H) by

$$u_H(x) := \begin{cases} u(x) & \text{if } x\in H,\\ \max\{u(x);u(\sigma x)\} & \text{if } x\in H^+,\\ \min\{u(x);u(\sigma x)\} & \text{if } x\in H^-. \end{cases} \tag{13}$$

Note that the two-point rearrangement is sometimes called *polarization* in the literature (see [4,8,9]).

Remark 1 The following properties are easy to check (see [1,2]): u_H is a rearrangement of u, and if $\operatorname{supp} u\subset B(R)$ for some $R>0$ then also $\operatorname{supp} u_H\subset B(R)$. Furthermore, if $u\in C(X)$ then we have $\omega_u(t)\geq\omega_{u_H}(t)$ $\forall t>0$. Finally, we have $(u^*)_H=u^*$, and if $u\in L^2(X,\mu)$, then by (8),

$$\|u_H-u^*\|_2\leq\|u-u^*\|_2.$$

3 INEQUALITY

Our main result is

THEOREM 1 *Let $X=\mathbf{R}^n,\mathcal{S}^n$ or $\mathbf{H}^n$, and let $F:(\mathbf{R}_0^+)^m\to\mathbf{R}$ continuous and satisfies* (10) *and in addition* (9) *in case that $X=\mathbf{R}^n$ or $\mathbf{H}^n$. Furthermore,* *let either*

(i) $u_1, \dots, u_m \in \mathcal{F}_+(X) \cap L^\infty(X, \mu)$ *or*
(ii) $u_1, \dots, u_m \in \mathcal{F}_+(X) \cap L^p(X, \mu)$ *and*

$$|F(v_1, \dots, v_m)| \leq C \sum_{i=1}^{m} v_i^p \quad \forall (v_1, \dots, v_m) \in (\mathbf{R}_0^+)^m, \tag{14}$$

for some $p \in [1, +\infty)$, $C > 0$.

In case (ii) *and if* $X = \mathbf{R}^n$ *or* $\mathbf{H}^n$ *we add the requirement that the functions* $u_1, \dots, u_m$ *have compact support. Then*

$$\int_X F(u_1, \dots, u_m)\, \mathrm{d}\mu(x) \leq \int_X F(u_1^*, \dots, u_m^*)\, \mathrm{d}\mu(x). \tag{15}$$

For the proof we need the following technical lemma which was also used in [12]. We include a proof for the convenience of the reader.

LEMMA 1 *Let* $F: (\mathbf{R}_0^+)^m \to \mathbf{R}$ *continuous and satisfies* (10). *Furthermore, let* $a_i, b_i, c_i^+, c_i^- \in \mathbf{R}_0^+$ *with*

$$c_i^+ = \max\{a_i; b_i\}, \quad c_i^- = \min\{a_i; b_i\}, \qquad i = 1, \dots, m.$$

Then

$$F(a_1, \dots, a_m) + F(b_1, \dots, b_m) \leq F(c_1^+, \dots, c_m^+) + F(c_1^-, \dots, c_m^-). \tag{16}$$

Proof W.l.o.g. we may assume that there is some $k \in \{1, \dots, m-1\}$ such that

$$a_i = c_i^+ \text{ and } b_i = c_i^- \quad \text{for } 1 \leq i \leq k.$$

Introducing the vectors $v' = (c_1^-, \dots, c_k^-)$, $v'' = (c_{k+1}^-, \dots, c_m^-)$, $h' = (h_1, \dots, h_k)$, $h'' = (h_{k+1}, \dots, h_m)$, where $h_i := c_i^+ - c_i^-$, $i = 1, \dots, m$, (16) reads as

$$I := F(v' + h', v'' + h'') + F(v', v'') - F(v' + h', v'') - F(v', v'' + h'') \geq 0. \tag{17}$$

Let $F \in C^2$. We have by Taylor's theorem,

$$I = \int_0^1 \left\{ \sum_{i=1}^{m} F_{v_i}(v' + th', v'' + th'')h_i - \sum_{i=1}^{k} F_{v_i}(v' + th', v'')h_i \right.$$
$$\left. - \sum_{i=k+1}^{m} F_{v_i}(v', v'' + th'')h_i \right\} \mathrm{d}t. \tag{18}$$

Now, since $F_{v_i v_j} \geq 0$ for $i \neq j$, we have for $t \in [0, 1]$,

$$F_{v_i}(v' + th', v'' + th'') - F_{v_i}(v' + th', v'')$$
$$= \int_0^1 \sum_{j=k+1}^{m} F_{v_i v_j}(v' + th', v'' + sth'')h_j \,\mathrm{d}s \geq 0, \quad \text{if } 1 \leq i \leq k,$$

and

$$F_{v_i}(v' + th', v'' + th'') - F_{v_i}(v', v'' + th'')$$
$$= \int_0^1 \sum_{j=1}^{k} F_{v_i v_j}(v' + sth', v'' + th'')h_j \,\mathrm{d}s \geq 0, \quad \text{if } k + 1 \leq i \leq m.$$

Therefore the integrand $\{\cdots\}$ in (18) is nonnegative, and (17) follows. In the general case we can argue by approximation.

Now we show that (11) holds for two-point rearrangements.

LEMMA 2 *Let* $F : (\mathbf{R}_0^+)^m \to \mathbf{R}$ *continuous and satisfies* (10). *Furthermore, let* $u_1, \ldots, u_m \in \mathcal{F}_+(X)$ *and* $H \in \mathcal{H}(X)$. *Then*

$$\int_X F(u_1, \ldots, u_m) \,\mathrm{d}\mu(x) \leq \int_X F((u_1)_H, \ldots, (u_m)_H) \,\mathrm{d}\mu(x). \tag{19}$$

Proof Let $Q(u_1, \ldots, u_m)$ denote the left integral in (19). Then we have by Lemma 1,

$$Q(u_1, \ldots, u_m)$$
$$= \int_{H^+} \{F(u_1(x), \ldots, u_m(x)) + F(u_1(\sigma x), \ldots, u_m(\sigma x))\} \,\mathrm{d}\mu(x)$$
$$\leq \int_{H^+} \{F((u_1)_H(x), \ldots, (u_m)_H(x))$$
$$+ F((u_1)_H(\sigma x), \ldots, (u_m)_H(\sigma x))\} \,\mathrm{d}\mu(x)$$
$$= Q((u_1)_H, \ldots, (u_m)_H).$$

Proof of Theorem 1 Our proof is much like the proof of Theorem 3 in [1]. Let Q as above. First we suppose that $u_i \in C(X)$, and if $X = \mathbf{R}^n$ or $\mathbf{H}^n$ then suppose in addition that u_i has compact support in $B(R)$ for some $R > 0$, $i = 1, \dots, m$. We define

$$\mathcal{S}(u_i) := \{U \in C(X)\colon \omega_U \leq \omega_{u_i} \text{ and } U \text{ is a rearrangement of } u_i\},$$
$$i = 1, \dots, m,$$

$$\mathcal{S}(u_1, \dots, u_m) = \{(U_1, \dots, U_m) \in \mathcal{S}(u_1) \times \cdots \times \mathcal{S}(u_m)\colon Q(U_1, \dots, U_m) \geq Q(u_1, \dots, u_m)\},$$

$$\delta = \inf\left\{ \sum_{i=1}^{m} \|U_i - u_i^*\|_2^2\colon (U_1, \dots, U_m) \in \mathcal{S}(u_1, \dots, u_m) \right\}.$$

If $X = \mathbf{R}^n$ or $\mathbf{H}^n$ then we add in the definition of $\mathcal{S}(u_i)$ the requirement that $\operatorname{supp} U \subset B(R)$. There exists $(U_1^0, \dots, U_m^0) \in \mathcal{S}(u_1, \dots, u_m)$ such that $\delta = \sum_{i=1}^m \|U_i^0 - u_i^*\|_2^2$. When $X = \mathcal{S}^n$ this follows from the theorem of Arzelá–Ascoli, and if $X = \mathbf{R}^n$ or $\mathbf{H}^n$ it follows from Arzelá–Ascoli together with the translation invariance of the integral $Q(u_1, \dots, u_m)$. If $\delta = 0$ then $(u_1^*, \dots, u_m^*) = (U_1^0, \dots, U_m^0) \in \mathcal{S}(u_1, \dots, u_m)$ and hence $Q(u_1^*, \dots, u_m^*) \geq Q(u_1, \dots, u_m)$, as required.

Suppose that $\delta > 0$. Then there exists $k \in \{1, \dots, m\}$ such that $U_k^0 \neq u_k^*$. It is easy to show (see [2]) that there exists $H \in \mathcal{H}(X)$ such that $\|(U_k^0)_H - u_k^*\|_2 < \|U_k^0 - u_k^*\|_2$, which means that

$$\sum_{i=1}^{m} \|(U_i^0)_H - u_i^*\|_2^2 < \sum_{i=1}^{m} \|U_i^0 - u_i^*\|_2^2.$$

Since also $((U_1^0)_H, \dots, (U_m^0)_H) \in \mathcal{S}(u_1, \dots, u_m)$ by Remark 1, this last inequality contradicts the definition of δ.

In the general cases we chosen sequences $u_1^{(k)}, \dots, u_m^{(k)}$ of nonnegative continuous functions with

$$\operatorname{supp} u_i^{(k)} \subset \operatorname{supp} u_i, \quad k = 1, 2, \dots,$$

and such that

$$u_i^{(k)} \longrightarrow u_i \quad \text{in } L^p(X, \mu), \tag{20}$$

in case (i), and

$$u_i^{(k)} \longrightarrow u_i \quad \text{in } L^1(X, \mu), \tag{21}$$

and

$$u_i^{(k)} \leq C, \quad \text{uniformly } \forall k \ \ (C > 0), \tag{22}$$

in case (ii), $i = 1, \ldots, m$; (20)–(22) also hold for $u_i^{(k)}, u_i$ replaced by $(u_i^{(k)})^*, u_i^*$, $k = 1, 2, \ldots$, $i = 1, \ldots, m$, respectivley, in view of (8). The assertion then follows from (14) and from Lebesgue's convergence theorem.

Remark 2 (1) Choosing $F(v_1, \ldots, v_m) = \prod_{i=1}^m v_i$ or $F(v_1, \ldots, v_m) = f(\sum_{i=1}^m v_i)$, where f is convex with $f(0) = 0$ we obtain the following inequalities which hold for nonnegative bounded functions $u_1, \ldots, u_m$ having compact support when $X = \mathbf{R}^n$ or $\mathbf{H}^n$:

$$\int_X \prod_{i=1}^m u_i \, \mathrm{d}\mu(x) \leq \int_X \prod_{i=1}^m u_i^* \, \mathrm{d}\mu(x), \tag{23}$$

$$\int_X f\left(\sum_{i=1}^m u_i\right) \mathrm{d}\mu(x) \leq \int_X f\left(\sum_{i=1}^m u_i^*\right) \mathrm{d}\mu(x). \tag{24}$$

(2) Theorem 1 implies analogous inequalities for the so-called (k, n)-*Steiner and cap symmetrizations* in $\mathbf{R}^n$, respectively S^n. Note that these symmetrizations can be seen as symmetric decreasing rearrangements on k-dimensional subspaces of X $(1 \leq k \leq n-1)$ (see [1]). For the proofs one can argue analogously as in [1, p. 59].

It is easy to obtain an inequality similar to (14) with F depending on x. Replace m by $m+1$ and let $u_{m+1} \in L^1(X, \mu) \cap L^\infty(X, \mu)$ smooth with $u_{m+1} = u_{m+1}^*$ and *strictly* radially decreasing. Defining a function $G : (\mathbf{R}_0^+)^{m+1} \to \mathbf{R}$ by the relation

$$G(v_1, \ldots, v_m, d(e, x)) := F(v_1, \ldots, v_m, u_{m+1}(x)),$$
$$x \in X \ \ (v_1, \ldots, v_m) \in (\mathbf{R}_0^+)^m,$$

we see that G is continuous and satisfies

$$G(v_1, \ldots, v_i + s, \ldots, v_m, z + t) - G(v_1, \ldots, v_i + s, \ldots, v_m, z)$$
$$- G(v_1, \ldots, v_i, \ldots, v_m, z + t) + G(v_1, \ldots, v_i, \ldots, v_m, z) \leq 0, \tag{25}$$

$$\begin{aligned} &G(v_1,\dots,v_i+s,\dots,v_j+t,\dots,v_m,z)\\ &\quad - G(v_1,\dots,v_i+s,\dots,v_j,\dots,v_m,z)\\ &\quad - G(v_1,\dots,v_i,\dots,v_j+t,\dots,v_m,z)\\ &\quad + G(v_1,\dots,v_i,\dots,v_j,\dots,v_m,z) \geq 0\\ &\qquad \forall s,t,v_1,\dots,v_m \in \mathbf{R}_0^+ \text{ and } \forall i,j \in \{1,\dots,m\},\ i \neq j. \end{aligned} \tag{26}$$

Note that if $G \in C^2$ then (25) and (26) imply

$$\frac{\partial^2 G}{\partial v_i \partial v_j} \geq 0, \quad \frac{\partial^2 G}{\partial v_i \partial z} \leq 0 \qquad \forall i,j \in \{1,\dots,m\}, \quad i \neq j.$$

The above considerations and Theorem 1 yield

COROLLARY 1 *Let* $X = \mathbf{R}^n$, $\mathcal{S}^n$ *or* $\mathbf{H}^n$. *Let* $G : (\mathbf{R}_0^+)^{m+1} \to \mathbf{R}$ *continuous with*

$$G(0,\dots,0,d(e,\cdot)) \in L^1(X,\mu), \tag{27}$$

and satisfying (25) *and* (26). *Furthermore, let either*

(i) $u_1,\dots,u_m \in \mathcal{F}_+(X) \cap L^\infty(X,\mu)$ *or*
(ii) $u_1,\dots,u_m \in \mathcal{F}_+(X) \cap L^p(X,\mu)$ *and*

$$|G(v_1,\dots,v_m,d(e,x))| \leq C \sum_{i=1}^m v_i^p + g(z) \quad \forall (v_1,\dots,v_m) \in (\mathbf{R}_0^+)^m, \tag{28}$$

for some $p \in [1,+\infty)$, $C > 0$ *and* $g \in L^1(X,\mu)$.

In case (ii) *and if* $X = \mathbf{R}^n$ *or* $\mathbf{H}^n$ *we add the requirement that the functions* $u_1,\dots,u_m$ *have compact support. Then*

$$\int_X G(u_1,\dots,u_m,d(e,x))\, \mathrm{d}\mu(x) \leq \int_X G(u_1^*,\dots,u_m^*,d(e,x))\, \mathrm{d}\mu(x). \tag{29}$$

Remark 3 Tahraoui [14] showed (29) in the special case that $X = \mathbf{R}^n$, $m = 2$ and $G \in C^3$. But his proof is quite complicated and it needs the unnecessary condition that $(\partial^3 G)/(\partial v_1 \partial v_2 \partial z) \leq 0$.

4 A SYMMETRY PROBLEM IN THE CALCULUS OF VARIATIONS

Consider the following variational problem:

$$J(u_1,\dots,u_m) \equiv \int_{B(R)} \left(\frac{1}{q}\sum_{i=1}^{m} |\nabla u_i|^q - G(u_1,\dots,u_m,|x|)\right) \mathrm{d}x \longrightarrow \text{Min.!},$$

$$u_i \in K := \{u \in W_0^{1,q}(B(R)) \colon u \geq 0\}, \quad i=1,\dots,m, \tag{30}$$

where $R>0$, $G : (\mathbf{R}_0^+)^{m+1} \to \mathbf{R}$ is continuous and satisfies (25)–(28) and $q \in (p,+\infty)$. Then it is easy to see that the functional J is bounded below and weakly lower semicontinuous. Hence there exists a minimizing solution. We prove

LEMMA 3 *There exists a solution of problem* (30) *with* $u_i = u_i^*$, $i=1,\dots,m$.

Proof Let $(U_1,\dots,U_m)$ a minimizing solution. By Corollary 1 we have

$$\int_{B(R)} G(U_1,\dots,U_m,|x|)\,\mathrm{d}x \leq \int_{B(R)} G(U_1^*,\dots,U_m^*,|x|)\,\mathrm{d}x. \tag{31}$$

Furthermore, there hold the following well-known inequalities (see e.g. [1]):

$$\int_{B(R)} |\nabla U_i|^q\,\mathrm{d}x \geq \int_{B(R)} |\nabla U_i^*|^q\,\mathrm{d}x, \quad i=1,\dots,m. \tag{32}$$

Now (31) and (32) yield $J(U_1^*,\dots,U_m^*) \leq J(U_1,\dots,U_m)$, and the assertion follows.

Remark 4 If $G_{v_i} \in C((\mathbf{R}_0^+)^{m+1})$ then a solution $(u_1,\dots,u_m)$ of problem (30) solves the following variational inequalities:

$$\int_{B(R)} |\nabla u_i|^{q-2}\nabla u_i \nabla(v-u_i)\,\mathrm{d}x \geq \int_{B(R)} G_{v_i}(u_1,\dots,u_m,|x|)(v-u_i)\,\mathrm{d}x$$

$$\forall v \in K, \ i=1,\dots,m. \tag{33}$$

Next suppose that $u_i>0$, $i=1,\dots,m$. (Note that this follows from the maximum principle in case that $G_{v_i}(u_1,\dots,u_m,|x|) \geq 0$ in $B(R)$,

$i = 1, \ldots, m$, for instance.) Then $(u_1, \ldots, u_m)$ is a weak solution of the following semilinear elliptic system,

$$\begin{aligned} -\Delta_q u_i \equiv -\nabla(|\nabla u_i|^{q-2}\nabla u_i) = G_{v_i}(u_1, \ldots, u_m, |x|) \text{in } B(R) \\ u_i = 0 \quad \text{on } \partial B(R), \quad i = 1, \ldots, m. \end{aligned} \tag{34}$$

The system is *cooperative* by (26). Systems of this form arise in modelling spatial phenomena in a variety of physical and chemical problems (see e.g. [5,6,15]).

It is worth to mention the following special case.

Let $q = 2$, and assume that the functions u_i and G_{v_i}, $i = 1, \ldots, m$ in (34) are smooth. Then the radial symmetry of u_i, $i = 1, \ldots, m$, follows as well via the *method of moving planes* (see [13,15]).

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