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Continuous Steiner-Symmetrization

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For nonnegative L-measurable functions $u: \mathbf{R}^n \to \mathbf{R}$ a continuous homotopy u^t, $0 \leq t \leq +\infty$, is constructed, connecting u with its Steinersymmetrization u^*. It is shown that a number of familiar relations between u and u^* including some integral inequalities are also valid for u and u^t. The method is applicable to prove symmetry properties of stationary solutions of variational problems.

1. Introduction

Consider a minimum problem of the form

$$(1) \qquad J(v) := \int_\Omega F(x, v, \nabla v)\, \mathrm{d}x \to \min!\,, \qquad v \in K\,,$$

where K is a convex subset of some functions space, e.g. $W_0^{1,p}(\Omega)$ or $L^p(\Omega)$, $p > 1$, and $\Omega \subseteq \mathbf{R}^n$ is a domain lying symmetrically to the hyperplane $\{y = 0\}$, $\{x = (x', y), x' \in \mathbf{R}^{n-1}, y \in \mathbf{R})$. If $v \in K$, we often also have $v^* \in K$, where v^* denotes the Steiner-symmetrization of v with respect to y, and

$$J(v^*) \leq J(v)\,.$$

It can be proved for the absolute minimum u of (1) that $u = u^*$.

This argumentation fails for local minima or stationary points w of the functional J. Therefore the following question is natural: Is there a (in the norm of X) continuous homotopy

$$t \mapsto v^t\,, \qquad 0 \leq t \leq +\infty\,, \qquad v^0 = v\,, \qquad v^\infty = v^*\,,$$

such that for $v \in K$ we also have $v^t \in K$ and

$$J(v^t) \leq J(v)\,,$$

or sharper:

(2) $$J(v) - J(v^t) \geq d\|v^t - v\|_X, \qquad d > 0, \qquad 0 \leq t \leq +\infty?$$

In this cases we could prove symmetry properties of w.

There are examples of such homotopies – called continuous symmetrization – in the literature. POLYA and SZEGÖ [4] considered nonnegative functions $v \in C(\mathbf{R}^n) \cap C^1(\Omega)$, where Ω is the compact support of v, which are quasiconcave in y, i.e., for any $x' \in \mathbf{R}^{n-1}$ and $c > 0$ the set $\{y : v(x', y) > c\}$ is convex. If we write $\{y : v(x', y) > c\} =: (y_1, y_2)$, the relations

(3) $$\{y : v^s(x', y) > c\} = (y_1^s, y_2^s), \qquad x' \in \mathbf{R}^{n-1}, \qquad c \geq 0,$$

with

(4) $$y_1^s := y_1 - \frac{s}{2}(y_1 + y_2),$$

$$y_2^s := y_2 - \frac{s}{2}(y_1 + y_2), \qquad 0 \leq s \leq 1,$$

define new functions $v^s \in C(\mathbf{R}^n) \cap C^1(\Omega^s)$ with compact support Ω^s, $0 \leq s \leq 1$, where the domains Ω^s again result from (3), (4) in the case $c = 0$.

Thus the mapping $s \mapsto v^s$, $0 \leq s \leq 1$, realizes a shift of the intervals $\{v(x', \cdot) > c\}$ in the y-direction with a "velocity" equally to the distance of their centre from the hyperplane $\{y = 0\}$ at the time $s = 0$, (see Figure 1). Obviously v^1 is the Steiner-symmetrization of v.

The authors proved then some integral inequalities. Thus the arc length and the outer and inner radius of a smooth convex plane curve and the capacity of a ringshaped plane domain bounded by two convex curves are diminished under continuous symmetrization.

BRASCAMP, LIEB and LUTTINGER [1] generalized the above construction for continuous functions v for which the sets $\{v(x', \cdot) > c\}$ consist only of a finite number of intervals. At the beginning the intervals of these level sets are shifted according to the rules (3), (4) up to a moment, in which two (or more) of them collide. At this moment the same procedure starts anew, where the rules (4) are to be applied on the already shifted set. This argument can be repeated again and again, such that after a finite number of steps the function v is transformed into its Steiner-symmetrization. The authors exploited this kind of continuous symmetrization together with Brunn's part of the Brunn-Minkowski-theorem to prove a

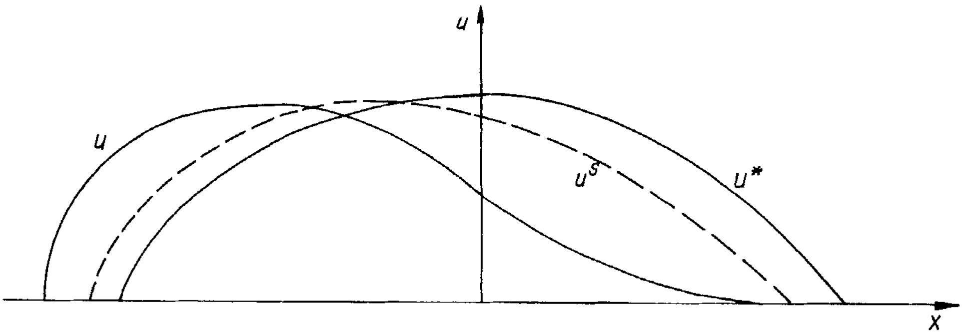

Fig. 1

very general convolution inequality for the cases of the Steiner- and Schwarz-symmetrizations.

Another variant of continuous symmetrization for functions, which are not quasiconcave, is considered by KAWOHL and MATANO [2, 3]. They proved symmetry properties of local minima of variational problems with various integral side conditions.

In this paper we describe a version of continuous symmetrization, which is in fact a generalization of the construction of [1] on L-measurable functions, and give its "L^p-theory". In Section 2 we introduce for L-measurable sets $M \subseteq \mathbf{R}^n$ a scale M^t, $0 \le t \le +\infty$, and prove some properties of the mapping $M \to M^t$, e.g. the monotonicity, the semigroup-property and the preservation of L-measure.

Then we define a scale v^t, $0 \le t \le +\infty$, corresponding to a L-measurable function $v: \mathbf{R}^n \to \mathbf{R}$, (Section 3). This is performed symmetrizing the level sets $\{v(x', \cdot) > c\}$, $x' \in \mathbf{R}^{n-1}$, $c \in \mathbf{R}$, continuously. The mapping $v \to v^t$, $0 \le t \le +\infty$, has a number of properties which are already known in the special case $t = +\infty$. Those are the equimeasurability, the nonexpansiveness in $L^p(\mathbf{R}^n)$ and some inequalities for product-integrals and convolutions of functions including the already mentioned general inequality from [1] (Section 4). We underline here that most of the proofs depend essentially on the monotonicity and the semigroup-property of the continuous symmetrization of sets and on an approximation argument with step-functions.

In Section 5 we prove the equicontinuity of the symmetrized functions which is useful for subsequent applications.

Section 6 deals with the common and different properties between our kind of symmetrization and the version of [2, 3].

In a later article we will prove inequalities of the sharp form (2), e.g., if J is the Dirichlet-integral, and investigate some applications on variational problems.

2. Continuous symmetrization of sets

We introduce some notations. For any set M in $\mathbf{R}^n$ we denote with $\chi(M)$ its characteristic function. Let $M(\mathbf{R}^n)$ be the whole of all L-measurable subsets of $\mathbf{R}^n$. If $M \in M(\mathbf{R}^n)$ we denote by $|M|$ its n-dimensional L-measure and by

$$S(M) = |M|^{-1} \int_M x \, dx$$

its centre of gravity if $|M| < +\infty$ and the above integral converges. We write $M \sim N$ if the sets $M, N \in M(\mathbf{R}^n)$ are equivalent, i.e., if we have $|M \setminus N| = |N \setminus M| = 0$.

Definition 1. (Symmetrization of L-measurable sets). Let $M \in M(\mathbf{R})$. Then

$$M^* := \left(-\frac{1}{2}|M|, +\frac{1}{2}|M| \right)$$

is called the symmetrization of M.

We have

$$(5) \qquad (M^*)^* = M^* .$$

According to (3) we now define the continuous symmetrization of open intervals in $\mathbf{R}$.

Definition 2.1. Let $M = (a, b)$, $-\infty < a < b < +\infty$, be a bounded open interval in $\mathbf{R}$. Then the scale of intervals $M^t := (a^t, b^t)$, $0 \le t \le +\infty$, where

$$(6) \qquad a^t := \frac{1}{2}\left(a - b + e^{-t}(a + b)\right),$$

$$b^t := \frac{1}{2}\left(b - a + e^{-t}(a + b)\right),$$

is called the *continuous symmetrization of M*.

We obtained the relations (6) from (4) by replacing $a = y_1$, $b = y_2$ and $t = -\log(1 - s)$. This yields the *semigroup-property*

$$(7) \qquad (M^s)^t = M^{s+t}, \qquad 0 \le s \le s + t \le +\infty,$$

which will be used later.

Note that in (6) we set $e^{-\infty} = 0$ and we take into account the usual rules

$$c + \infty = +\infty + c = \infty,$$

for $c \ge 0$ in (7).

Next we extend the notion of continuous symmetrization on open sets in $\mathbf{R}$.

Definition 2.2. Let $M \subseteq \mathbf{R}$ be an open set. Then there are scales of sets $M(t)$, $0 \le t \le +\infty$, whose elements $M(t)$ have the following properties:

(i) $M(0) = M$,

and

(ii) if I is some bounded open interval with $I \subseteq M(s)$, $s \in [0, +\infty]$, then also $I^t \subseteq M(s + t)$ for any $t \in [0, +\infty]$.

For any $t \in [0, +\infty]$ we introduce a set

$$\mathscr{E}(t) := \{M(t) : M(t) \text{ satisfies (i), (ii)}\}.$$

The the scale of sets M^t, $0 \le t \le +\infty$, **defined by the relations**

$$M^t := \bigcap_{M(t) \in \mathscr{E}(t)} M(t),$$

is called the *continuous symmetrization of M*.

Note that the Definitions 2.1 and 2.2 are consistent. Later we shall show that M^t is open if M is open (see (18)).

From Definition 2.2 it follows that the scale M^t, $0 \le t \le +\infty$, satisfies the semigroup-property (7), (i) and (ii) and the following monotonicity-property:

$$(8) \qquad \text{If} \quad M \subseteq N, \quad \text{then} \quad M^t \subseteq N^t \quad \text{for any} \quad t \in [0, +\infty].$$

For any $t \in [0, +\infty]$ and for any arbitrary two open sets M and N in $\mathbf{R}$, it follows from (8) immediately that

$$(9) \qquad (M \cap N)^t \subseteq M^t \cap N^t$$

and

(10) $$(M \cup N)^t \supseteq M^t \cup N^t .$$

Now we show how simple sets are to be continuously symmetrized.

Remark 1 (Continuous symmetrization of open sets).

1. (Crucial case). Let M be a finite union of pairwise disjoint bounded open intervals:

(11) $$M = \bigcup_{k=1}^{m} I_k .$$

Taking $s = 0$ and $I = I_k$, $k = 1, \ldots, m$, in (ii) it follows

(12) $$M^t \supseteq \bigcup_{k=1}^{m} (I_k)^t , \qquad 0 \leq t \leq +\infty .$$

The intervals $(I_k)^t$ are disjoint for t less than or equal to

$$t_1 := \min \left\{ \log \frac{2\,|S(I_j) - S(I_k)|}{|I_j| + |I_k|} : 1 \leq j, k \leq m \right\},$$

and for $t = t_1$ some of them meet each other in their endpoints. Further, the relation (12) is valid with the equality sign for $0 \leq t \leq t_1$ because of the minimality of the sets M^t.

Since $M^t = (M^{t_1})^{t - t_1}$, we can argue analogously for parameters $t \geq t_1$. Thus we get by induction numbers

$$m =: m_0 > m_1 > \ldots > m_{N-1} := 1 ,$$
$$0 =: t_0 < t_1 < \ldots < t_N := +\infty ,$$

and bounded open intervals $I_{k,l}$, $k = 1, \ldots, m_l$, such that for any $t \in (t_l, t_{l+1}]$, $l = 0, \ldots, N-1$, we have

(13) $$M^t = \bigcup_{k=1}^{m_l} (I_{k,l})^{t - t_l} ,$$

where the intervals $(I_{k,l})^{t-t_l}$ are pairwise disjoint, some of them meet each other in their endpoints for $t = t_{l+1}$, and

(14) $$\lim_{t \uparrow t_{l+1}} |M^{t_{l+1}} \setminus M^t| = 0 , \qquad l = 0, \ldots, N-1 .$$

Further it follows that

(15) $$|M^t| = |M| , \qquad 0 \leq t \leq +\infty ,$$

(preservation of L-measure),

and

$$M^\infty = \left(-\frac{1}{2} |M|, + \frac{1}{2} |M| \right).$$

2. Let $M = (a, +\infty)$, $a \in \mathbf{R}$. For $b > a$ we have then

$$(a, b) =: M_b \subset M\,, \quad \text{and} \quad (M_b)^t \subset M^t\,, \qquad t > 0\,.$$

A short computation shows that for any $t > 0$ we get

$$\bigcup_{n=1}^{+\infty} (M_n)^t = \mathbf{R}\,,$$

and therefore also

$$M^t = \mathbf{R}\,, \qquad 0 < t \leq +\infty\,. \tag{16}$$

Similarly, (16) can be shown if $M = (-\infty, b)$, $b \in \mathbf{R}$, or if M is any open set containing an unbounded interval.

3. Let M be any open set. Then we have

$$M = \bigcup_{k=1}^{+\infty} I_k = \bigcup_{n=1}^{+\infty} M_n\,, \quad \text{where} \quad M_n = \bigcup_{k=1}^{n} I_k\,, \qquad n = 1, 2, \ldots, \tag{17}$$

and the I_k's are pairwise disjoint open intervals. It follows that

$$(M_n)^t \subseteq (M_{n+1})^t \subset M^t\,, \qquad n = 1, 2, \ldots, 0 \leq t \leq +\infty\,,$$

and in view of the minimality of the sets M^t we have that

$$M^t = \bigcup_{n=1}^{+\infty} (M_n)^t\,, \qquad 0 \leq t \leq +\infty\,, \tag{18}$$

i.e., the symmetrized sets M^t, $0 \leq t \leq +\infty$, are open too. If M has finite measure, then (15) holds. Otherwise we can conclude that

$$M^\infty = \mathbf{R}\,. \tag{19}$$

Now we introduce a continuous symmetrization of L-measurable sets in $\mathbf{R}$. The symmetrized sets are unique up to nullsets. This will be sufficient for our purposes.

Remark 2. Let $M \in M(\mathbf{R})$. Then we have a representation

$$M = \bigcap_{n=1}^{+\infty} O_n \setminus N\,, \tag{20}$$

where $O_n \supseteq O_{n+1}$, $n = 1, 2, \ldots$, are open sets and N is a nullset. We observe that for any $t \in [0, +\infty]$ the set

$$M(t) := \bigcap_{n=1}^{+\infty} O_n^t \tag{21}$$

is unique except for a nullset, i.e., it depends not on the special representation (20). In fact, let

$$M = \bigcap_{n=1}^{+\infty} O_n' \setminus N'$$

be any other representation of M of the form (20) and

$$M(t)' = \bigcap_{n=1}^{+\infty} (O'_n)^t ,$$

then we have

$$(O_n \cup O'_n)^t \supseteq (O_n)^t \cup (O'_n)^t \supseteq (O_n)^t \cap (O'_n)^t \supseteq (O_n \cap O'_n)^t , \qquad n = 1, 2, \ldots,$$

from which we easily conclude that $M(t) \sim M(t)'$.

This gives rise to the following

Definition 2.3. Let M be any set in $M(\mathbf{R})$ which is not open. Then any scale of sets M^t, $0 \leq t \leq +\infty$, with

$$(22) \qquad M^t \sim M(t),$$

where the sets $M(t)$ are defined by (20) and (21), is called a continuous symmetrization of M.

We can immediately conclude:

Theorem 1. *Let $M \in M(\mathbf{R})$. Then* (15) *is valid. Further there is a version M^t, $0 \leq t \leq +\infty$, such that relations* (7) – (10) *hold and*

$$(23) \qquad M^\infty = M^* .$$

Remark 3. Let $M, N \in M(\mathbf{R})$. Then for any $t \in [0, +\infty]$ we have by (9) that

$$M^t \setminus N^t = M^t \setminus (M^t \cap N^t) \subseteq M^t \setminus (M \cap N)^t .$$

Since $|M \setminus N| = |M^t \setminus (M \cap N)^t|$, we conclude the following relation (for later reference):

$$(24) \qquad |M^t \setminus N^t| \leq |M \setminus N| , \qquad 0 \leq t \leq +\infty .$$

Remark 4. Let M be as in Remark 1, case 1. Then a simple computation shows that

$$S(M^t) = e^{-t} S(M) \quad \text{for any} \quad t \in [0, +\infty] .$$

But by approximation (25) follows also in the case that $M \in M(\mathbf{R})$ and that $S(M)$ exists.

Now we prove some continuity properties of the symmetrization.

Theorem 2 (Continuity of the mapping $M \to M^t$). *Let $M, M_n \in M(\mathbf{R})$, $n = 1, 2, \ldots$ Then if*

$$(26) \qquad M_n \to M \quad \textit{in measure} ,$$

it follows that

$$(27) \qquad (M_n)^t \to M^t \quad \textit{in measure and uniformly in} \quad 0 \leq t \leq +\infty .$$

Further if

$$(28) \qquad \chi(M_n) \to 1 \quad \textit{a.e.} \quad \textit{in } M ,$$

then for any $t \in [0, +\infty]$

$$(29) \qquad \chi(M_n^t) \to 1 \quad \textit{a.e.} \quad \textit{in } M^t .$$

If, moreover, there is a set $E \in M(\mathbf{R})$ *of finite L-measure, such that*

(30) $$M, M_n \subseteq E, \qquad n = 1, 2, \dots,$$
$$\chi(M_n) \to \chi(M) \quad \text{a.e.} \quad \text{in } E,$$

then for any $t \in [0, \pm\infty]$

(31) $$\chi(M_n^t) \to \chi(M^t) \quad \text{a.e.} \quad \text{in } E^t.$$

Proof. Assume first (26). Then assertion (27) follows from the inequalities

$$|M_n^t \setminus M^t| \le |M_n \setminus M| \quad \text{and} \quad |M^t \setminus M_n^t| \le |M \setminus M_n|.$$

If the sequence M_n suffices (28) then by Egorov's Theorem ([6], p. 108) for any $\delta > 0$ there is a measurable subset M_δ of M with $|M \setminus M_\delta| \le \delta$ such that the convergence (28) is uniformly on M_δ. Therefore we have $M_\delta \subseteq M_n$ for sufficiently large n. Because of (8) there are versions M^t, M_n^t, such that $M_\delta^t \subseteq M_n^t$ for the above n and also $M_\delta^t \subseteq M^t$, $0 \le t \le +\infty$. Taking $\delta \to 0$ it follows (29).

Next we assume (30) and introduce the sets

$$N_n := \bigcup_{k=n}^{+\infty} M_n, \qquad n = 1, 2, \dots,$$

For any $t \in [0, +\infty]$ we have

$$N_n^t \supseteq N_{n+1}^t \supseteq M^t, \qquad n = 1, 2, \dots,$$

and

$$\lim_{n \to \infty} \int_{E^t \setminus M^t} \chi(N_n^t) = 0$$

because of (29). Since the sequence $\chi(N_n^t)$, $n = 1, 2, \dots$, is nonnegative and monotone decreasing it follows that

(32) $$\chi(N_n^t) \to 0 \quad \text{a.e. in } E^t \setminus M^t.$$

Now from (29) and (32) we conclude (31).

In general we cannot symmetrize sets of infinite measure continuously, which can be seen from Remark 1, case 2. This makes the restrictions in the following theorem intelligible.

Theorem 3 (Continuity of the mapping $t \mapsto M^t$). *Let* $M \in M(\mathbf{R})$ *have a finite L-measure and let* t_n, $n = 1, 2, \dots$, *be any sequence converging to a number* $t \in [0, +\infty]$. *Then*

(33) $$M^{t_n} \to M^t \quad \text{in measure}.$$

If the set M is open, then

(34) $$\chi(M^{t_n}) \to 1 \quad \text{a.e.} \quad \text{in } M^t,$$

and if moreover the set

(35) $$E := \bigcup_{n=1}^{+\infty} M^{t_n}$$

has finite L-measure, then

(36) $\chi(M^{t_n}) \to \chi(M^t)$ a.e. *in* $\mathbf{R}$.

Proof. Let first M be an open set with finite measure and take $\varepsilon > 0$ small. Since M has a representation of the form (17), we get $|M \setminus M_m| < \frac{\varepsilon}{2}$ with $m = m(\varepsilon)$ large enough.

By the earlier considerations, the sets $(M_m)^{t_n}$ consist of no more than m open intervals with a length greater than or equal to $\min\{|I_k| : k = 1, \dots, m\}$, and the boundaries of the intervals depend continuously on the parameter t_n. Therefore we can find a set N_ε such that for sufficiently large n we have

$$N_\varepsilon \subseteq M_m^{t_n} \quad \text{and} \quad |M_m^{t_n} \setminus N_\varepsilon| < \frac{\varepsilon}{2}.$$

Thus we get $|M^{t_n} \setminus N_\varepsilon| < \varepsilon$ for these n, which proves (34).

Now assume (35). Setting

$$N_n := \bigcup_{k=n}^{+\infty} M^{t_n},$$

we get

$$E \supseteq N_n \supseteq N_{n+1} \supseteq M^t, \qquad n = 1, 2, \dots,$$

and

$$\lim_{n \to +\infty} \int_{E \setminus M^t} \chi(N_n) = 0.$$

Since the sequence $\chi(N_n)$, $n = 1, 2, \dots$, is nonnegative and monotone decreasing, it follows that

$$\chi(N_n) \to 0 \quad \text{a.e.} \quad \text{in } E \setminus M^t,$$

which proves (36).

Now take $M \in M(\mathbf{R})$ with $|M| < +\infty$. Then the representations (20) and (21) are valid. For a given $\varepsilon > 0$ we take m large enough such that $|O_m \setminus M| < \frac{\varepsilon}{2}$ and then n_0 large enough such that for any $n \geq n_0$

$$|O_m^{t_n}| - |O_m^t \cap O_m^{t_n}| < \frac{\varepsilon}{2}$$

holds. Then it follows that

$$|M^t| - |M^t \cap M^{t_n} = |O_m^t| - |O_m^t \cap O_m^{t_n}| - |O_m^t \setminus M^t| + |(O_m^t \cap O_m^{t_n}) \setminus (M^t \cap M^{t_n})|$$
$$< \frac{\varepsilon}{2} + |O_m^{t_n} \setminus M^{t_n}| < \varepsilon,$$

which proves (33).

The following lemma will be useful to prove continuity properties of symmetrized functions.

Lemma 1. *Let M, N be open sets with $M \subseteq N$. Then*

$$\operatorname{dist}\{M; \partial N\} \le \operatorname{dist}\{M^t; \partial N^t\}, \qquad 0 \le t \le +\infty. \tag{37}$$

Proof: Assume that $N = \bigcup_{i=1}^{m} J_i$ for pairwise disjoint open intervals J_i. Then we have that $M = \bigcup_{i=1}^{m} M_i$ for open sets $M_i \subseteq J_i$, $i = 1, \dots, m$. If we denote by $I_i(t)$ and I_i the smallest open intervals containing M_i^t and M_i, respectively, then a short computation shows that $I_i(t) \subseteq I_i^t$ and

$$\operatorname{dist}\{I_i; \partial J_i\} \le \operatorname{dist}\{I_i^t; \partial J_i^t\}, \qquad i = 1, \dots, m, \qquad 0 \le t \le +\infty.$$

If t_1 is the largest number $t > 0$ for which the intervals J_i^t, $i = 1, \dots, m$, remain disjoint, we get for $0 \le t < t_1$ that

$$\begin{aligned}\operatorname{dist}\{M; \partial N\} &= \min\{\operatorname{dist}\{M_i; \partial J_i\} : i = 1, \dots, m\}\\ &= \min\{\operatorname{dist}\{I_i; \partial J_i\} : i = 1, \dots, m\}\\ &\le \min\{\operatorname{dist}\{I_i^t; \partial J_j^t\} : i = 1, \dots, m\}\\ &\le \min\{\operatorname{dist}\{M_i^t; \partial J_i^t\} : i = 1, \dots, m\} = \operatorname{dist}\{M^t; \partial N^t\}.\end{aligned}$$

The value $\operatorname{dist}\{M^t; \partial N^t\}$ may have a jump of positive magnitude crossing $t = t_1$. Now because of the semigroup-property (7) the assertion (37) follows for any $t \in [0, +\infty]$.

If M and N are open sets the assertion follows by approximation.

Now we introduce the continuous symmetrization of sets in $M(\mathbf{R}^n)$, $n \ge 2$. Let $M \subseteq \mathbf{R}^n$. We set

$$M' := \{x' \in \mathbf{R}^{n-1} : (x', y) \in M,\ y \in \mathbf{R}\}, \qquad \text{(the projection of } M \text{ on } \mathbf{R}^{n-1})$$

and

$$M(x') := \{y \in \mathbf{R} : (x', y) \in M,\ x' \in M'\}, \qquad \text{(the intersection of } M \text{ with } (x', \mathbf{R})).$$

Note that any set $M \subseteq M(\mathbf{R}^n)$ has the representation

$$M = \{x = (x', y) : y \in M(x'),\ x' \in M'\}$$

where $M' \in \mathbf{R}^{n-1}$ and for almost any $x' \in \mathbf{R}^{n-1}$ $M(x') \in M(\mathbf{R})$. If, moreover, M is open, then the sets $M(x')$ are open in $\mathbf{R}$ for any $x' \in \mathbf{R}^{n-1}$.

Definition 3. Let $M \in M(\mathbf{R}^n)$. Then the set

$$M^* := \left\{x = (x', y) : -\frac{1}{2}|M(x')| < y < \frac{1}{2}|M(x')|,\ x' \in M'\right\}$$

is called the *(Steiner-)symmetrization of M with respect to y.*

Definition 4.1. Let M be an open set in $\mathbf{R}^n$. Then the scale of sets M^t, $0 \leq t \leq +\infty$, with

$$M^t = \{x = (x', y): y \in (M(x'))^t, x' \in M'\},$$

where the sets $(M(x'))^t$ are defined by Definition 2.2, is called the *continuous symmetrization* of M.

Definition 4.2. Let $M \in M(\mathbf{R}^n)$ and suppose that M is not open. Then any scale of sets M^t, $0 \leq t \leq +\infty$, with

$$M^t \sim \{x = (x', y): y \in (M(x'))\,(t), x' \in M'\},$$

where the sets $(M(x'))(t)$ are defined by (20), (21), is called a *continuous (Steiner-)-symmetrization of M with respect to y.*

Theorem 4. *Theorems 1, 2 and 3, Lemma 1 and Remark 2 remain valid for the continuous symmetrization of sets in* $M(\mathbf{R}^n)$, $n \geq 2$.

Proof. We want to show (37) in the case $n \geq 2$. Let $M \subseteq N \subseteq \mathbf{R}^n$. Then for any $t \in [0, +\infty]$ we have that

$$[\operatorname{dist}\{M^t; \partial N^t\}]^2$$
$$= \min\{|x' - \xi'|^2 + (\operatorname{dist}\{(M(x'))^t; \partial(N(\xi'))^t\})^2 : x' \in M', \xi' \in N'\}.$$

Because of Pythagoras' Theorem the above minima are attained for two values $x', \xi' \in \mathbf{R}^{n-1}$ with $(M(x'))^t \subseteq (N(\xi'))^t$ (see Figure 2). But if $x', \xi' \in \mathbf{R}^{n-1}$ are any two values with $M(x') \subseteq N(\xi')$, we have by Lemma 1 that

$$\operatorname{dist}\{M(x'); \partial N(\xi')\} \leq \operatorname{dist}\{(M(x'))^t; \partial(N(\xi'))^t\}, \qquad 0 \leq t \leq +\infty,$$

and (37) follows.

The proof of the other properties are simple and therefore we leave them to the reader.

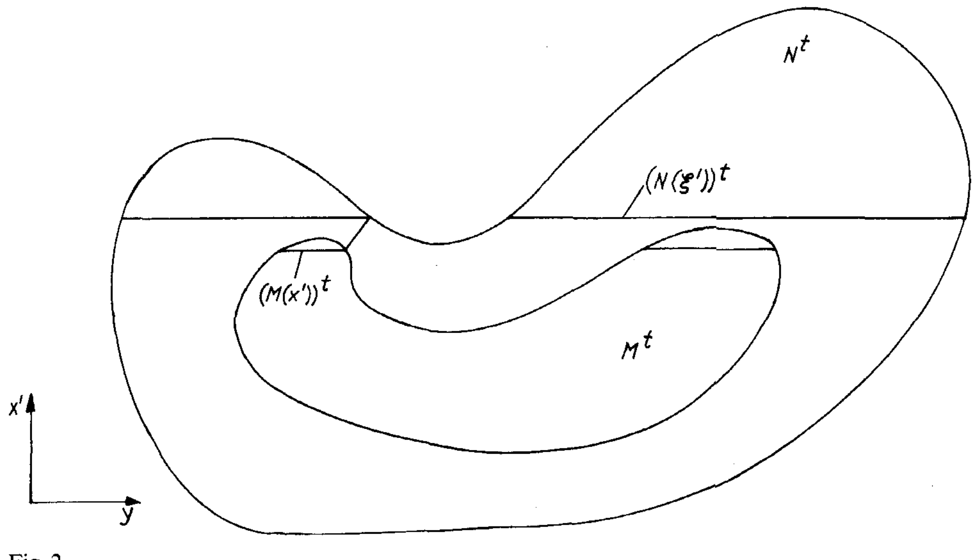

Fig. 2

For a set $M \in M(\mathbf{R}^n)$ with finite measure we can define the coordinates of the centre of gravity by

$$S_{x_i}(M) := |M|^{-1} \int_M x_i \, dx, \qquad i = 1, \dots, n-1,$$

$$S_y(M) := |M|^{-1} \int_M y \, dx,$$

if the above integrals converge.

Remark 5. Let $M \in M(\mathbf{R}^n)$ and let the numbers $S_y(M)$, $S_{x_i}(M)$, $i = 1, \dots, n-1$, exist. Then

(38) $$S_{x_i}(M^t) = S_{x_i}(M), \qquad i = 1, \dots, n-1,$$

(39) $$S_y(M^t) = e^{-t} S_y(M), \qquad 0 \le t \le +\infty.$$

3. Continuous symmetrization of functions

We consider function $f : \mathbf{R}^n \to \mathbf{R}$ with the following property:

(40) There is a number $c_f \in [-\infty, +\infty)$, such that $f(x) \ge c_f$ on $\mathbf{R}^n$, and for any $c > c_f$ the level sets $\{x \in \mathbf{R} : f(x) > c\}$ have finite L-measure.

We denote by $F(\mathbf{R}^n)$ the set of functions satisfying (40). If $f, g \in F(\mathbf{R}^n)$ and $f = g$ a.e., then we write $f \sim g$. Frequently we use the subclass $F^+(\mathbf{R}^n)$ of $F(\mathbf{R}^n)$ given by the functions having the number c_f in (40) equal to zero. We remark that the class $F(\mathbf{R}^n)$ is large enough to contain functions f for which the set $\{f < c_f\}$ has infinite measure. If $f, g \in F(\mathbf{R}^n)$ are bounded below, we get $(f + g) \in F(\mathbf{R}^n)$, and if $\lambda > 0$ then $(\lambda f) \in F(\mathbf{R}^n)$. Further if we define a function ${}_cf$ by

$${}_cf(x) := \max\{f(x); c\}, \qquad x \in \mathbf{R}^n, \qquad c \in \mathbf{R},$$

then also ${}_cf \in F(\mathbf{R}^n)$.

Finally if f is continuous, then the level sets $\{f > c\}$ are open for any $c \in \mathbf{R}$.

Definition 5. Let $f \in F(\mathbf{R})$ and let

$$m_f(c) := |\{x \in \mathbf{R} : f(x) > c\}|, \qquad c > c_f,$$

be the corresponding distribution function of f. We set

$$x = X(c) := \frac{1}{2} m_f(c), \qquad c > c_f.$$

Then the inverse function $f^* \in F(\mathbf{R})$ satisfying the relations

$$c = f^*(x) = f^*(-x), \qquad c > c_f,$$

is called the symmetrization of f.

Definition 6. Let $f \in F(\mathbf{R}^n)$, $n \geq 2$. Then for almost any $x' \in \mathbf{R}^{n-1}$ there exists the distribution function

$$m_f(x', c) := |\{y : f(x', y) > c\}|, \qquad c > c_f,$$

and the function $y = Y(x', c) := \frac{1}{2} m_f(x', c)$ has an inverse function f^*

$$c = f^*(x', y) = f^*(x', -y), \qquad c > c_f.$$

f^* is called the *(Steiner-)symmetrization of f with respect to y.*

Definition 7.1. Let $f \in F(\mathbf{R}^n)$ be continuous. Then the scale of functions f^t, $0 \leq t \leq +\infty$, defined by all the relations

$$\{f^t > c\} := \{f > c\}^t, \qquad c > c_f,$$
$$\{f^t = c_f\} := \mathbf{R}^n \setminus \bigcup_{c > c_f} \{f > c\}^t,$$

is called the *continuous (Steiner-)symmetrization of f with respect to y* in the case $n \geq 2$, and the *continuous symmetrization of f in the case $n = 1$.*

Definition 7.2. Let $f \in F(\mathbf{R}^n)$ be not continuous. Then any scale of functions

$$f^t, 0 \leq t \leq +\infty,$$

defined by all the relations

$$\{f^t > c\} \sim \{f > c\}^t, \qquad c > c_f, \tag{41}$$
$$\{f^t = c_f\} \sim \mathbf{R}^n \setminus \bigcup_{c > c_f} \{f > c\}^t,$$
$$\{f^t = +\infty\} \sim \bigcap_{c > c_f} \{f > c\}^t,$$

is called a *continuous (Steiner-)symmetrization of f with respect to y* in the case $n \geq 2$, and a *continuous symmetrization of f* in the case $n = 1$.

We remark that in the right-hand sides of (41) we used arbitrary scales of level sets $\{f > c\}^t$. Therefore the functions f^t are unique except for the nullsets.

If f is a step-function (see Figure 3a–c), then there are sets $M_i \in M(\mathbf{R}^n)$ with $M_i \supseteq M_{i+1}$, $i = 1, \ldots, m-1$, and numbers $c_0 \in \mathbf{R}$, $c_i \in \mathbf{R}^+$, $i = 1, \ldots, n$, such that

$$f = c_0 + \sum_{i=1}^{m} c_i \chi(M_i). \tag{42}$$

Then we have $M_1^t \supseteq \ldots \supseteq M_m^t$ and because of (41)

$$f^t = c_0 + \sum_{i=1}^{m} c_i \chi(M_i^t), \qquad 0 \leq t \leq +\infty. \tag{43}$$

Any function $f \in F(\mathbf{R}^n)$ can be approximated in measure by step-functions of the form (42), where the sets M_i may consist of finitely many intervals only, and where all the c^i coincide.

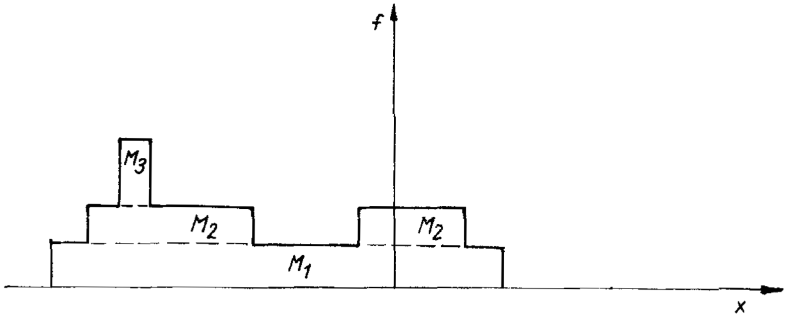

Fig. 3a

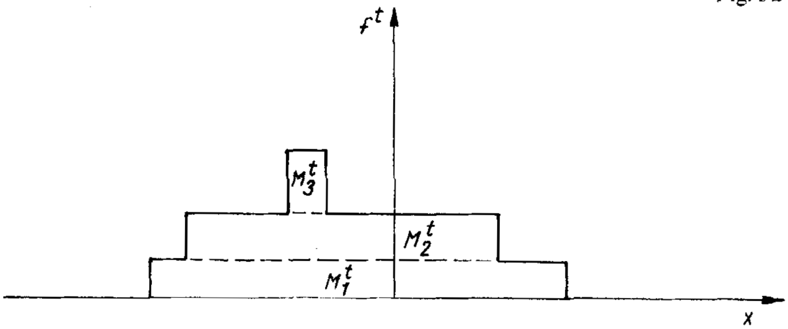

Fig. 3b

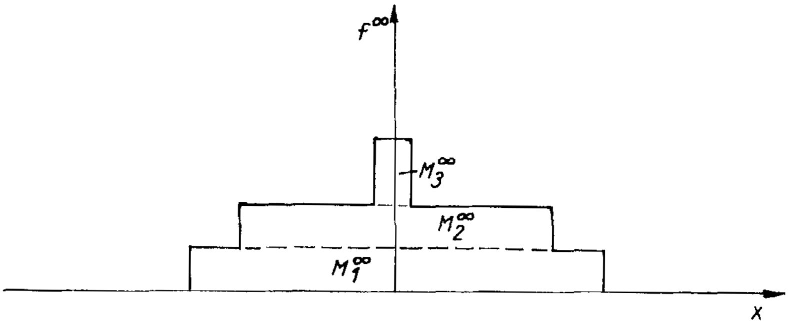

Fig. 3c

The following properties are easy applications of the results of the previous section on the level sets of symmetrized functions.

Theorem 5. *Let* $f, g \in F(\mathbf{R}^n)$, $\lambda > 0$, $d \in \mathbf{R}$, $c_f < a < b$, $c_f < c$, $0 \leq t \leq t + s \leq +\infty$. *Then*

(44) $$(f + d)^t \sim f^t + d\,;$$

(45) $$(\lambda f)^t \sim \lambda f^t\,;$$

(46) $$\{f \geq c\}^t \sim \{f^t \geq c\}\,;$$

(47) $$\left|\{a \leq f < b\}\right| = \left|\{a \leq f^t < b\}\right|, \qquad \textit{(preservation of L-measure)}\,;$$

(48) $f^t \le g^t$ a.e. if $f \le g$ a.e., *(monotonicity)*;

(49) $f^{t+s} \sim (f^t)^s$, *(semigroup-property)*;

(50) $({}_cf)^t \sim {}_c(f^t)$.

We introduce now the centre $S(f)$ of functions $f \in F^+(\mathbf{R}^n)$. In the case $n = 1$ we set

$$S(f) := \left(\int_{\mathbf{R}} f(x)\,\mathrm{d}x\right)^{-1} \int_{\mathbf{R}} x f(x)\,\mathrm{d}x,$$

and if $n \ge 2$ we set

$$S(f) := (S_{x_1}(f), \ldots, S_{x_{n-1}}(f), S_y(f))$$

with

$$S_{x^i}(f) := \left(\int_{\mathbf{R}^n} f(x)\,\mathrm{d}x\right)^{-1} \int_{\mathbf{R}^n} x_i f(x)\,\mathrm{d}x, \qquad i = 1, \ldots, n-1,$$

$$S_y(f) := \left(\int_{\mathbf{R}^n} f(x)\,\mathrm{d}x\right)^{-1} \int_{\mathbf{R}^n} y f(x)\,\mathrm{d}x,$$

if the above integrals converge. $S(f)$ is the centre of gravity of $\mathbf{R}^n$ with mass density $f(x)$.

Remark 6. Let $f \in F(\mathbf{R}^n)$, $t \in [0, +\infty]$ and suppose that $S(f)$ exists. Then we get in the case $n = 1$ that

(51) $S(f^t) = \mathrm{e}^{-t} S(f)$,

and if $n \ge 2$

(52) $S(f^t) = (S_{x_1}(f), \ldots, S_{x_{n-1}}(f), \mathrm{e}^{-t} S_y(f))$.

Proof. The relations (51) and (52) follow from Remark 5 for step-functions by an easy calculation and then for functions $f \in F(\mathbf{R}^n)$ by approximation.

According with Theorems 2 and 3 we have the following continuity properties:

Theorem 6 (Continuity of the mapping $f \to f^t$). *Let $f, f_n \in F(\mathbf{R}^n)$, $n = 1, 2, \ldots$, be a.e. finite functions and assume that there is a set $E \in M(\mathbf{R}^n)$ of finite measure and a number $c_0 \in \mathbf{R}$ such that*

(53) $f \ge c_0$, $f_n \ge c_0$ *in* $\mathbf{R}^n$, $\{f > c_0\} \subseteq E$, $\{f_n > c_0\} \subseteq E$, $n = 1, 2, \ldots$.

If

(54) $f_n \to f$ *in measure*,

then

(55) $f_n^t \to f^t$ *in measure and uniformly in* $t \in [0, +\infty]$.

If in addition

(56) $f_n \to f$ *a.e. in* E,

then we have for any $t \in [0, +\infty]$ that

(57) $f_n^t \to f^t$ *a.e. in* E^t.

Proof. Assume that (53), (54) are satisfied and let $\delta > 0$. We introduce the sets

$$M_{in}(t) := \left\{c_0 + i\frac{\delta}{2} < f_n^t \le c_0 + (i+1)\frac{\delta}{2}\right\} \cap \left\{c_0 + (i+2)\frac{\delta}{2} < f^t\right\},$$

$$N_{in} := \left\{c_0 + i\frac{\delta}{2} < f_n \le c_0 + (i+1)\frac{\delta}{2}\right\}, \qquad i = 0, 1, \ldots, \quad n = 1, 2, \ldots.$$

From (47) it follows that

$$|M_{in}(t)| \le |N_{in}|, \qquad i = 0, 1, \ldots, \qquad n = 1, 2, \ldots,$$

and

$$|\{f^t - f_n^t \ge \delta\}| \le \sum_{i=0}^{+\infty} |M_{in}(t)| \le \sum_{i=0}^{+\infty} |N_{in}| \le |E|.$$

Further we conclude from (24) and (54) that for any $i = 0, 1, \ldots,$

$$|M_{in}(t)| \le \left|\left\{c_0 + (i+1)\frac{\delta}{2} < f^t\right\} \setminus \left\{c_0 + (i+1)\frac{\delta}{2} < f_n^t\right\}\right|$$
$$\le \left|\left\{c_0 + (i+1)\frac{\delta}{2} < f\right\} \setminus \left\{c_0 + (i+1)\frac{\delta}{2} < f_n\right\}\right| \to 0 \quad \text{as} \quad n \to +\infty.$$

This shows also that $|\{f^t - f_n^t \ge \delta\}| \to 0$ as $n \to +\infty$. Together with an analogue consideration for the sets $\{f_n^t - f^t \ge \delta\}$ we derive (55).

Now assume (53) and (56). For any $c > c_0$ we set

$$M := \{f > c\}, \qquad M_n := \{f_n > c\}, \qquad n = 1, 2, \ldots.$$

Then we get from Theorem 2 that for any $t \in [0, +\infty]$

$$\chi(M_n^t) \to \chi(M^t) \quad \text{a.e.} \quad \text{in } E^t.$$

Since c was arbitrary, the assertion (57) follows.

Theorem 7 (Continuity of the mapping $t \mapsto f^t$). *Let* t_n, $n = 1, 2, \ldots$, *be any sequence converging to* $t \in [0, +\infty]$ *and let* $f \in \mathrm{F}(\mathbf{R}^n)$ *be a.e. finite and bounded below. Then*

(58) $f^{t_n} \to f^t$ *in measure*.

If, moreover, f is continuous and the set

(59) $\bigcup_{n=1}^{+\infty} \{f^{t_n} > c_f\}$

has finite L-measure, (with the number c_f defined by (40)), then

(60) $$f^{t_n} \to f^t \quad \text{everywhere in } \mathbf{R}^n .$$

Proof. Since f is bounded below, we conclude that c_f is finite. We choose some number $\delta > 0$ and introduce the sets

$$M_{in}(t) := \left\{c_f + i\frac{\delta}{2} < f^{t_n} \le c_f + (i+1)\frac{\delta}{2}\right\} \cap \left\{c_f + (i+2)\frac{\delta}{2} < f^t\right\},$$

$$N_i := \left\{c_f + i\frac{\delta}{2} < f^t \le c_f + (i+1)\frac{\delta}{2}\right\}, \qquad i = 0, 1, \ldots, \quad n = 1, 2, \ldots .$$

From (47) it follows that $|M_{in}| \le |N_i|$, $i = 0, 1, \ldots, n = 1, 2, \ldots$, and

$$|\{f^t - f^{t_n} \ge \delta\}| \le \sum_{i=0}^{+\infty} |M_{in}| \le |\{c_f + \delta < f^t\}| + \sum_{i=1}^{+\infty} |N_i| < +\infty .$$

Further we conclude from (33) that

$$|M_{in}| \le \left|\left\{c_0 + (i+1)\frac{\delta}{2} < f^t\right\} \Big\backslash \left\{c_f + (i+1)\frac{\delta}{2} < f^{t_n}\right\}\right| \to 0 \quad \text{as} \quad n \to +\infty .$$

This shows also that $|\{f^t - f^{t_n} \ge \delta\}| \to 0$ as $n \to +\infty$. Together with an analogue consideration for the sets $\{f^{t_n} - f^t \ge \delta\}$ we derive (58).

Now we assume that (59) holds and that f is continuous. Then the level sets

$$\{f^t > c\}, \{f^{t_n} > c\}, \qquad n = 1, 2, \ldots, \qquad c > c_f,$$

are open, and the assertion (60) follows from (36).

Now we show some integral relations between the symmetrized functions which are already known for the special cases $t = +\infty$.

Theorem 8. *Let $t \in [0, +\infty]$, let φ and ψ be continuous functions with φ monotone increasing. If $f \in F(\mathbf{R}^n)$, then*

(61) $$(\varphi(f))^t \sim \varphi(f^t) .$$

If $\psi(f) \in L^1(\mathbf{R}^n)$, *then*

(62) $$\int_{\mathbf{R}^n} \psi(f^t(x))\,\mathrm{d}x = \int_{\mathbf{R}^n} \psi(f(x))\,\mathrm{d}x .$$

The mapping $f \to \int_{\mathbf{R}^n} f^t(x)\,\mathrm{d}x$ is additive, i.e.,

(63) $$\int_{\mathbf{R}^n} (f+g)^t(x)\,\mathrm{d}x = \int_{\mathbf{R}^n} (f^t(x) + g^t(x))\,\mathrm{d}x , \qquad f, g \in L^1(\mathbf{R}^n) .$$

The mapping $f \to f^t$ is nonexpansive in $L^1(\mathbf{R}^n)$, i.e.,

(64) $$\|f^t - g^t\|_{L^1(\mathbf{R}^n)} \le \|f - g\|_{L^1(\mathbf{R}^n)} , \qquad f, g \in L^1(\mathbf{R}^n) .$$

Proof. (61)–(63) follow from the preservation of measure (47). Now we prove (64). First let f, g be step functions of the form

$$f = c + \varepsilon \sum_{i=1}^{m} \chi(F_i), \qquad g = c + \varepsilon \sum_{i=1}^{m} \chi(G_i), \tag{65}$$

for measurable sets $F_i \supseteq F_{i+1}$, $G_i \supseteq G_{i+1}$, $i = 1, \dots, m-1$, $\varepsilon > 0$, $c \in \mathbf{R}$. Because of Theorem 1 we have that for any $t \in [0, +\infty]$

$$\begin{aligned}\int_{\mathbf{R}^n} |f^t(x) - g^t(x)| \,\mathrm{d}x &= \varepsilon \sum_{i=1}^{m} (|F_i^t| + |G_i^t| - 2\,|F_i^t \cap G_i^t|) \\ &\le \varepsilon \sum_{i=1}^{m} (|F_i| + |G_i| - 2\,|F_i \cap G_i|) = \int_{\mathbf{R}^n} |f(x) - g(x)| \,\mathrm{d}x .\end{aligned}$$

If $f, g \in F(\mathbf{R}^n)$, then (64) follows by approximation.

The following property is crucial for further considerations.

Lemma 2. *Let $f, g \in F(\mathbf{R}^n)$, $c \ge 0$ and $0 \le t \le +\infty$. Then*

$$\int_{\{f^t > g^t\}} (f^t(x) - g^t(x)) \,\mathrm{d}x \le \int_{\{f > g\}} (f(x) - g(x)) \,\mathrm{d}x , \tag{66}$$

and

$$\int_{\{|f^t - g^t| > c\}} (|f^t(x) - g^t(x)| - c) \,\mathrm{d}x \le \int_{\{|f - g| > c\}} (|f(x) - g(x)| - c) \,\mathrm{d}x , \tag{67}$$

if the integrals of the right side in (66) *and* (67) *converge.*

Proof. Assume that (66) is false, i.e.,

$$\int_{\{f^t > g^t\}} (f^t(x) - g^t(x)) \,\mathrm{d}x > \int_{\{f > g\}} (f(x) - g(x)) \,\mathrm{d}x . \tag{68}$$

Then because of (62) it follows that

$$\int_{\{f^t \le g^t\}} (f^t(x) - g^t(x)) \,\mathrm{d}x \quad \int_{\{f \le g\}} (f(x) - g(x)) \,\mathrm{d}x ,$$

which together with (68) yields

$$\int_{\mathbf{R}^n} |f^t(x) - g^t(x)| \,\mathrm{d}x > \int_{\mathbf{R}^n} |f(x) - g(x)| \,\mathrm{d}x ,$$

contradicting (64).

Next by means of (44) and (66) we get the inequalities

$$\int_{\{f^t > g^t + c\}} (f^t(x) - g^t(x) - c) \,\mathrm{d}x \le \int_{\{f > g + c\}} (f(x) - g(x) - c) \,\mathrm{d}x$$

and

$$\int_{\{g^t > f^t + c\}} (g^t(x) - f^t(x) - c) \,\mathrm{d}x \le \int_{\{g > f + c\}} (g(x) - f(x) - c) \,\mathrm{d}x .$$

The addition of both inequalities yields (67).

More general is the following result:

Theorem 9. *Let $j:\mathbf{R}_0^+ \to \mathbf{R}_0^+$ be a continuous and convex function with $j(0) = 0$, and assume that $f, g \in F(\mathbf{R}^n)$, $j(|f - g|) \in L^1(\mathbf{R}^n)$. Then for any $t \in [0, +\infty]$*

$$\int_{\mathbf{R}^n} j(|f^t(x) - g^t(x)|)\,dx \le \int_{\mathbf{R}^n} j(|f(x) - g(x)|)\,dx. \tag{69}$$

Proof. We define a function $h = h(c, z)$, $c \ge 0$, $z \ge 0$, by

$$h(c, z) := \begin{cases} 0 & \text{if } 0 \le z \le c, \\ z - c & \text{if } c \le z. \end{cases}$$

Therefore instead of (69) we have

$$\int_{\mathbf{R}^n} h(c, |f^t(x) - g^t(x)|)\,dx \le \int_{\mathbf{R}^n} h(c, |f(x) - g(x)|)\,dx. \tag{70}$$

Now assume that $j \in C^2$ with $j'(0) = 0$. Then

$$j(z) = \int_0^{+\infty} \lambda(c)\, h(c, z)\,dc, \qquad z \ge 0, \tag{71}$$

with $\lambda(z) = j''(z) \ge 0$. Then the assertion follows from (70) and (71) by superposition.

Further, if $j \in C^2$ we can write $j(z) = \alpha z + j_1(z)$ with $\alpha \ge 0$ and $j_1'(0) = 0$. Then (69) follows from (64) and the above considerations.

Finally, if j is any continuous and convex function with $j(0) = 0$, the assertion follows by approximation with convex C^2-functions.

Lemma 3 (Nonexpansivity of the mapping $f \to f^t$ in $L^p(\mathbf{R}^n)$). *Let $f, g \in F(\mathbf{R}^n) \cap L^p(\mathbf{R}^n)$, $1 \le p \le +\infty$. Then for any $t \in [0, +\infty]$*

$$\|f^t - g^t\|_{L^p(\mathbf{R}^n)} \le \|f - g\|_{L^p(\mathbf{R}^n)}. \tag{72}$$

Proof. For $1 \le p < +\infty$ (72) follows from (69) taking $j(z) = z^p$ and in the limit case $p = +\infty$ by approximation.

The following lemma is well known in the case $t = +\infty$ (see e.g. [7]) and is a special case of Theorem 10 in the following section. We give here an independent proof.

Lemma 4 (Hardy-Littlewood-inequality). *Let $f, g \in F(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$. Then for any $t \in [0, +\infty]$*

$$\int_{\mathbf{R}^n} f^t(x)\, g^t(x)\,dx \ge \int_{\mathbf{R}^n} f(x)\, g(x)\,dx. \tag{73}$$

Proof. We have by (72) and (62) that

$$\begin{aligned}\int_{\mathbf{R}^n} f^t(x)\, g^t(x)\,dx &= \frac{1}{2}(\|f^t\|_{L^2}^2 + \|g^t\|_{L^2}^2 - \|f^t - g^t\|_{L^2}^2) \\ &\ge \frac{1}{2}(\|f\|_{L^2}^2 + \|g\|_{L^2}^2 - \|f - g\|_{L^2}^2) = \int_{\mathbf{R}^n} f(x)\, g(x)\,dx.\end{aligned}$$

4. Inequalities for product-integrals and convolutions

There are many examples of rearrangement inequalities for product-integrals and convolutions in the literature. Possibly the most general inequality of this kind appeared in [1] (see the remarks in the introduction). Since the authors already used the continuous symmetrization for simple cases, it is not surprising that the inequality can be extended on the whole scale of continuous Steiner-symmetrizations.

Theorems 10. *Let $f_i \in F^+(\mathbf{R}^n)$, $a_{ij} \in \mathbf{R}$, $1 \le i \le k$, $1 \le j \le l$. Then for any $t \in [0, +\infty]$*

$$\int_{\mathbf{R}^n} \dots \int_{\mathbf{R}^n} \prod_{i=1}^{k} f_i\left(\sum_{j=1}^{l} a_{ij}x_j\right) \mathrm{d}x_1 \dots \mathrm{d}x_l \le \int_{\mathbf{R}^n} \dots \int_{\mathbf{R}^n} \prod_{i=1}^{k} f_i^t\left(\sum_{j=1}^{l} a_{ij}x_j\right) \mathrm{d}x_1 \dots \mathrm{d}x_l . \tag{74}$$

Remark 7. Theorem 10 is nontrivial only for $k > l$. If $k < l$ or if $k = l$ and $\det(a_{ij}) = 0$, then both integrals in (74) diverge. If $k = l$ and $\det(a_{ij}) \neq 0$, then equality holds. $\Big($This can be seen, if we change the variables to $y_i = \sum_{j=1}^{l} a_{ij}x_j$ and then use the fact that $\int f_i = \int f_i^t$.$\Big)$

From (74) we can derive by specialization some rearrangement inequalities for product-integrals and convolutions which are well-known for the Steiner-symmetrization in the case $t = +\infty$ and for other kinds of rearrangements, e.g., the Schwarz-symmetrization.

Corollary 1. *Let $u_i \in F^+(\mathbf{R}^n)$, $i = 1, \dots, m$, $m \ge 2$. Then for any $t \in [0, +\infty]$*

$$\int_{\mathbf{R}^n} \prod_{i=1}^{m} u_i^t(x)\, \mathrm{d}x \ge \int_{\mathbf{R}^n} \prod_{i=1}^{m} u_i(x)\, \mathrm{d}x , \tag{75}$$

if one of the integrals in (75) converges.

Corollary 2. *Let $f, g, h \in F^+(\mathbf{R}^n)$. Then for any $t \in [0, +\infty]$*

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f^t(x)\, g^t(y)\, h^t(x - y)\, \mathrm{d}x\, \mathrm{d}x \ge \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f(x)\, g(y)\, h(x - y)\, \mathrm{d}x\, \mathrm{d}x , \tag{76}$$

if one of the integrals in (76) converges.

Proof of Theorem 10. First we observe that by Fubini's theorem it suffices to show (74) in the case $n = 1$. Further, as in the proof of Theorem 8, we can restrict ourselves to the case that the f_i's are step functions with a finite number of values. Then, after an easy calculation, (74) reduces to the case that the f_i's are characteristic functions of measurable sets. Finally, by Theorem 7, it suffices to prove this inequality for the case that these measurable sets are finite unions bounded intervals. But for this we refer to the elegant proof in [1], Lemma 2.1., which makes use of Brunn's part of the Brunn-Minkowski-Theorem. We note here that, except for some change of the parameter t, the authors in [1] used the same kind of continuous symmetrization as in our article. Thus the theorem is proved.

5. Continuity of the symmetrized functions

Lemma 1 is sufficient to prove continuity properties of the symmetrized functions.

Theorem 11. *Let $f \in F^+(\mathbf{R}^n)$ be continuous. Then the functions f^t, $0 \leq t \leq +\infty$, are equicontinuous.*

Further, if f is Lipschitz-continuous with Lipschitz-constant L, then the functions f^t, $0 \leq t \leq +\infty$, are also Lipschitz-continuous with the same constant L.

Proof. Taking $M^t = \{f^t > c_2\}$, $N^t = \{f^t > c_1\}$, $0 < c_1 < c_2$, $0 \leq t \leq +\infty$, in (38), we get for any $\varepsilon > 0$ and $0 \leq t \leq +\infty$ that

$$\min\{|x_1 - x_2| : |f(x_1) - f(x_2)| \geq \varepsilon\} \leq \min\{|x_1 - x_2| : |f^t(x_1) - f^t(x_2)| \geq \varepsilon\},$$

and the assertion follows.

Remark 8. The Lipschitz-continuity is in fact the "best" regularity which holds under continuous symmetrization of functions. This can be seen by symmetrizing a function $f \in C^1(\mathbf{R})$ which is not quasiconcave.

The functions f^t and f^∞ in Figure 4b and 4c are no longer differentiable in the marked points.

6. Another variant of continuous symmetrization

In the following we give another definition of continuous symmetrization which is due to Kawohl and Matano ([2, 3]).

Definition 2.1'. Let $M = (a, b)$, $-\infty < a < b < +\infty$, be a bounded open interval in $\mathbf{R}$. Then the scale of intervals $M^t := (a^t, b^t)$, $0 \leq t \leq +\infty$, where

$$a^t := \begin{cases} a - t & \text{if } 0 \leq t \leq \frac{1}{2}(a+b), \\ a + t & \text{if } 0 \leq t \leq -\frac{1}{2}(a+b), \\ \frac{1}{2}(a-b) & \text{if } 0 < \frac{1}{2}|a+b| < t, \end{cases}$$

(77)

$$b^t := \begin{cases} b - t & \text{if } 0 \leq t \leq \frac{1}{2}(a+b), \\ b + t & \text{if } 0 \leq t \leq -\frac{1}{2}(a+b), \\ \frac{1}{2}(b-a) & \text{if } 0 < \frac{1}{2}|a+b| < t, \end{cases}$$

is called the *continuous symmetrization II of M*.

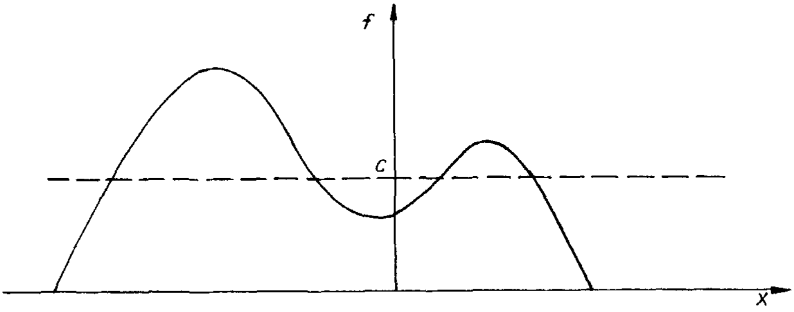

Fig. 4a

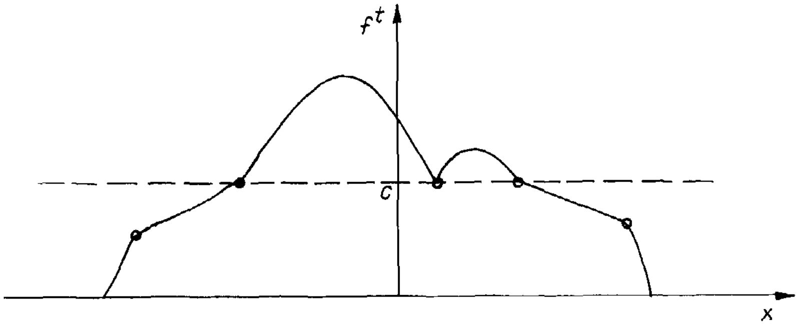

Fig. 4b

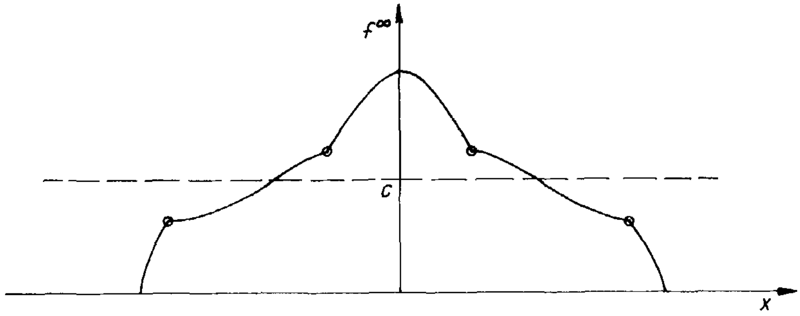

Fig. 4c

The formulas (77) again yield the semigroup-property (7). Therefore we can define a new continuous symmetrization II for L-measurable sets as in Section 2, and for functions as in Section 4 only by replacing the formulas (6) by (77). Most of the earlier proved properties hold with identical proofs. In the following we give the main differences.

If M is a union of pairwise disjoint bounded open intervals given by formula (11), then (12) follows, and the intervals $(I_k)^t$ are disjoint for

$$t \leq t_1 := \min\left\{\max\{|S(I_j)|;\ |S(I_k)|\} - \frac{1}{2}(|I_j| + |I_k|) : j, k = 1, \dots, m, j \neq k\right\}.$$

Further we again can conclude the formulas (13)–(15), where the number t_N is *finite*. More generally, if $M \in M(\mathbf{R}^n)$ is a bounded set, we obtain that $M^t \sim M^*$ for $t \geq t^*$ for some finite number t^*.

If M is an unbounded open interval, then (16) is not true but (19) remains valid.

Let $M \in M(\mathbf{R}^n)$, $n \geq 2$. Then the formulas (38) remain true. It is easy to show that the number $S_y(M^t)$ tends monotonically with t to zero, but we cannot find a simple formula like (39).

Similar results are obtained for the centre of sets in $\mathbf{R}$ and for the centre of functions.

Remark 9. Let in (1) the functional J be the Dirichlet-integral

$$J(v) := \int_{-a}^{+a} (v')^2(x)\,dx\,,$$

with $K := \{v \in W_0^1([-a, +a]),\ v \geq 0\}$, $a > 0$.

We want to compute $J(u^t)$, $0 \leq t \leq +\infty$, for quasiconcave functions

$$u \in K \cap C^2([-a, +a])$$

in the case of the continuous symmetrization II.

1. Consider first a function u with $u'(\pm a) = 0$, and assume that the centres of the level sets $\{u > c\}$ are different positive numbers for any $c > 0$, (Figure 5).

Then for any $t \in (0, +\infty]$ there is a number $c(t) > 0$, such that u^t is symmetric with respect to 0 for values less than or equal to $c(t)$ and the graph of u^t is a congruent shift to the left of the graph of u for values greater than or equal to $c(t)$. A computation shows that $\lim_{t \to 0} \frac{1}{t} c(t) = 0$. Therefore we get $J(u^t) - J(u) = o(t)$ for a small $t > 0$, i.e., we have not a sharp estimate of the form (2).

2. Next let u be a function of the following form.

There is a number $c_0 > 0$ such that for any $c \in [0, c_0]$ the level sets $\{u > c\}$ are symmetric with respect to 0. Further u has a "plateau" at the value c_0, i.e., $|\{u = c_0\}| > 0$, and for $c > c_0$ the centres of the level sets $\{u > c\}$ are different positive numbers, (Figure 6).

Then it is easy that the values $J(u^t)$ remain unchanged for a small $t > 0$.

The above described effects were removed in [2] in the following way.

In Example 1.: let $d > 0$ and $u(\alpha) = u(\beta) = d$, $-a < \alpha < \beta < a$. Then if we first continuously symmetrize the graph of u only for values $u \geq d$ with respect to the axis $x = \frac{1}{2}(\alpha + \beta)$, we can derive for a small $t > 0$ an estimate of the form (2). The same is

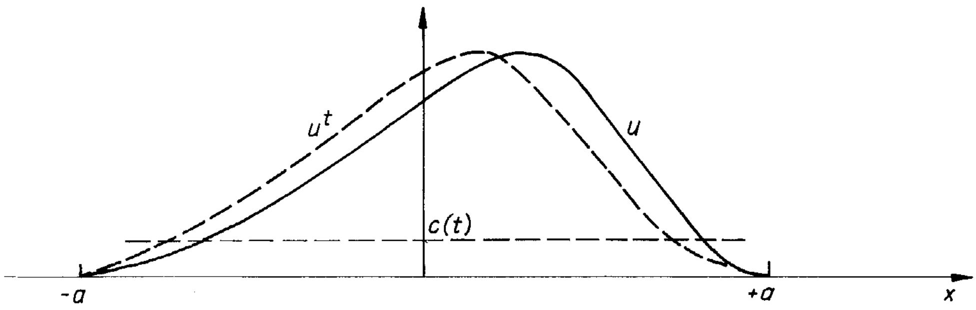

Fig. 5

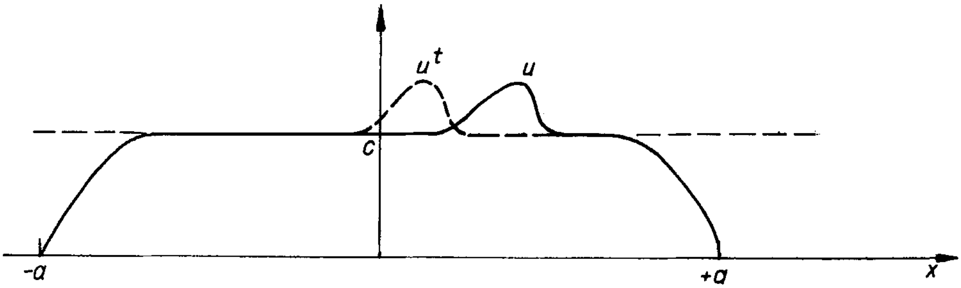

Fig. 6

true in Example **2.** if we take $d > c$. However, this idea seems not to be applicable for dimensions $n \geq 2$.

This was the motivation for the author to introduce a new continuous symmetrization where the described pathologies do not occur. Indeed, using the continuous symmetrization based on Definition 2.1, we can obtain the sharp estimate (2) in both cases **1.** and **2.** The reason is that all the nonvanishing derivatives of u are already changed from the first moment $t = 0$ on.

In a further paper we will prove inequalities of the form (2) where the integrand F contains derivatives of the symmetrized functions.

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